

Green tensor of the Stokes system and asymptotics of stationary Navier-Stokes flows in the half space

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Abstract

We derive refined estimates of the Green tensor of the stationary Stokes system in the half space. We then investigate the spatial asymptotics of stationary solutions of the incompressible Navier-Stokes equations in the half space. We also discuss the asymptotics of fast decaying flows in the whole space and exterior domains. In the Appendix we consider axisymmetric self-similar solutions.

Keywords: Navier-Stokes equations; Stokes system; half space, exterior domain; Green tensor; Odqvist tensor; spatial asymptotics; asymptotic profile; asymptotic completeness; self-similar solutions.

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1 Introduction

We are concerned with the Stokes system in the n -dimensional half space \mathbb{R}_+^n , $n \geq 2$,

$$\begin{cases} -\Delta u + \nabla q = f + \nabla \cdot F, & \operatorname{div} u = 0 & \text{in } \mathbb{R}_+^n, \\ u = 0 & & \text{on } \partial\mathbb{R}_+^n, \end{cases} \quad (\text{S})$$

or of the Navier-Stokes equations

$$\begin{cases} -\Delta u + (u \cdot \nabla)u + \nabla p = f + \nabla \cdot F, & \operatorname{div} u = 0 & \text{in } \mathbb{R}_+^n, \\ u = 0 & & \text{on } \partial\mathbb{R}_+^n. \end{cases} \quad (\text{NS})$$

Above $u = (u_i)_{i=1}^n : \mathbb{R}_+^n \rightarrow \mathbb{R}^n$ is the velocity field, $p : \mathbb{R}_+^n \rightarrow \mathbb{R}$ is the pressure, and $(f + \nabla \cdot F)_i = f_i + \partial_j F_{ji}$ is the given force. We denote

$$\mathbb{R}_+^n = \{x = (x', x_n) : x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}, x_n > 0\}, \quad (1.1)$$

with boundary $\Sigma = \partial\mathbb{R}_+^n = \{x = (x', x_n) : x' \in \mathbb{R}^{n-1}, x_n = 0\}$. Denote

$$x^* = (x', -x_n) \quad \text{if } x = (x', x_n). \quad (1.2)$$

The purpose of this paper is to study the asymptotic behavior of the Navier-Stokes flows for small forces. To this end, we also derive pointwise estimates of the Green tensor for the Stokes system (S). Our linear results are valid for dimension $n \geq 2$, while our nonlinear results are mostly for $n \geq 3$.

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1.1 Background and motivation

As shown by Lorentz [11] (see also [16, 5], §2.1), the fundamental solution $\{U_{ij}(x)\}_{i,j=1,\dots,n}$ of the Stokes system in the whole space \mathbb{R}^n has the same decay properties as that for the Laplace equation, namely (for $n \geq 3$)

$$|U_{ij}(x)| \lesssim |x|^{2-n}. \quad (1.3)$$

(We denote $A \lesssim B$ if there is some constant C so that $A \leq CB$.) As a result, when the force is small (of order ϵ) and sufficiently localized (i.e. the force decays sufficiently fast), one can construct the solutions to the Navier-Stokes equations with the same decay

$$|u_i(x)| \lesssim \epsilon \langle x \rangle^{2-n}, \quad \langle x \rangle := (2 + |x|^2)^{1/2}. \quad (1.4)$$

By a standard cut-off argument, one can get solutions with the same decay in an exterior domain (see [4]).

However, when the domain is the half space \mathbb{R}_+^n with no-slip boundary condition, the Green tensor $\{G_{ij}(x, y)\}_{i,j=1,\dots,n}$ to (S) has a faster decay rate than (1.3),

$$|G_{ij}(x, y)| \lesssim |x|^{1-n}, \quad (|y| \leq 1 \ll |x|), \quad (1.5)$$

(see Section 2 for detailed review), and one can construct solutions to (NS) with the same decay (see e.g. [2], [5])

$$|u_i(x)| \lesssim \epsilon \langle x \rangle^{1-n} \quad (1.6)$$

for small localized forces.

This project starts with the following intuition: For fixed $|y| \lesssim 1$ (corresponding to localized force), the decay of $G_{ij}(x, y)$ in x should be similar to the Poisson kernel of (S). It has been shown by Odqvist [15, §2] (see §2.2) that the Poisson tensor of (S) is

$$K_{ij}(x) = \frac{2x_n x_i x_j}{\omega_n |x|^{n+2}}, \quad (1.7)$$

where $\omega_n = \frac{2\pi^{n/2}}{n\Gamma(n/2)}$ is the volume of the unit ball in \mathbb{R}^n . Thus we expect that

$$|G_{ij}(x, y)| \lesssim \frac{x_n}{|x|^n}, \quad (|y| \lesssim 1 \ll |x|). \quad (1.8)$$

For $x_n \sim |x|$, this estimate reduces to (1.5), while it implies more decay than (1.5) for $x_n \ll |x|$. As a result, the Navier-Stokes flow for a small localized force is expected to have the same decay as the Green tensor. The goal of this paper is justify this intuition and identify the leading asymptotic profile of solutions of (NS) with small localized force.

1.2 Main results

Section 2 is concerned with the refined upper bounds for the Green tensor and its derivatives of the Stokes system in \mathbb{R}_+^n for $n \geq 3$ and $n = 2$. In particular, when $n \geq 3$, for $x, y \in \mathbb{R}_+^n$ we have

$$|G_{ij}(x, y)| \leq \frac{C_0 x_n y_n}{|x - y|^{n-2} |x - y^*|^2}, \quad i, j \in \{1, \dots, n\}, \quad (1.9)$$

where the constant $C_0 > 0$ is independent of $x, y \in \mathbb{R}_+^n$, and recall $y^* = (y', -y_n)$ for $y = (y', y_n)$. Furthermore, when $j = n$, the estimate (1.9) can be improved as

$$|G_{in}(x, y)| \leq \frac{C_0 x_n y_n^2}{|x - y|^{n-2} |x - y^*|^3}. \quad (1.10)$$

The above estimates justify (1.8) and imply extra decay when $j = n$ and $|y| \ll |x|$. See Theorems 2.4 for the above estimates, and (1.13) and Theorem 2.5 for refined gradient estimates.

In Section 3, we identify the leading profile of the Navier-Stokes flows in \mathbb{R}_+^n , $n \geq 3$, for small localized forces. To be more precise, suppose that $|f(x)| \lesssim \varepsilon \langle x \rangle^{-a}$ and $|F(x)| \lesssim \varepsilon \langle x \rangle^{-a+1}$ with $a \in (n+1, n+2)$ for sufficiently small $\varepsilon > 0$. Then, there exists a unique solution (u, p) of the Navier-Stokes equations (NS) with $|u(x)| \lesssim \frac{\varepsilon x_n}{\langle x \rangle^n}$ and, furthermore, its asymptotics is given as

$$u_i(x) = \sum_{j=1}^{n-1} K_{ij}(x) \tilde{b}_j + O\left(\frac{\varepsilon x_n}{\langle x \rangle^{a-1}}\right), \quad (1.11)$$

where

$$\tilde{b}_j = \int_{\mathbb{R}_+^n} \{u_n(y)u_j(y) + y_n f_j(y) - F_{nj}(y)\} dy, \quad (j < n). \quad (1.12)$$

Here, for simplicity, we assume that $a \in (n+1, n+2)$ but it suffices to restrict $a > n+1$ (see Theorem 3.6 for the details). On the other hand, for any given small numbers $\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_{n-1}$ we construct a solution of the Navier-Stokes equations satisfying (1.11) and (1.12) (consult Theorem 3.7). For the Stokes system, we present similar formulas including two dimension for fast decaying f and F without smallness assumption (see Theorem 3.4).

In vector form, with $(\vec{K}_j)_i = K_{ij}$, (1.11) reads $u(x) = \sum_{j=1}^{n-1} \vec{K}_j(x) \tilde{b}_j + \text{error}$. Thus the leading asymptotic of the solution is given by a linear combination of $\vec{K}_1, \dots, \vec{K}_{n-1}$. That \vec{K}_n is not present is because a solution of (3.1)-(3.2) should have zero flux on any hemisphere $S_R^+ = \{x \in \mathbb{R}_+^n, |x| = R\}$, while \vec{K}_n has nonzero flux.

To derive (1.11), it is required to estimate the derivatives of the Green tensor. However it is not an easy task, as the formulas for the Green tensor span more than one full page in the literature (see [13, Appendix 1] for $n = 2, 3$, and [5, IV.3] for higher dimensions). Fortunately, we are able to refine the approach of [13, Appendix 1] and derive estimates for derivatives of G_{ij} for $n \geq 2$,

$$\left| \nabla_x^\alpha \nabla_y^\beta G_{ij}(x, y) \right| \leq \frac{C_m x_n}{|x - y|^{n-2+m} |x - y^*|} \quad (1.13)$$

for any multi-indices α and β with $|\alpha| + |\beta| = m > 0$ and $\alpha_n = 0$ (see Theorem 2.5). We emphasize that the factor x_n in (1.13) is lost only if $\alpha_n > 0$ and differentiations in the y variable does not kill the x_n factor in (1.13). This is important for the refined error estimates, which contain the x_n factor, in (1.11) and Theorem 3.6.

As applications, we consider the asymptotics of *general* solutions in \mathbb{R}_+^n in Theorem 3.8 under various smallness assumptions on the forces or the solutions, and we also consider similar questions when we further remove the boundary condition in a neighborhood of the origin in Theorem 3.9. The latter turns out to be a type of aperture problem and we recover previously known asymptotic profiles of solutions with a refined decay estimate for error terms (see Theorem 3.9 for the details and compare with [1] and [5]).

In Section 4, we extend the methods of Section 3 and study the asymptotics of *fast decaying* solutions of the Navier-Stokes equations in the *whole space and exterior domains* in \mathbb{R}^n , $n \geq 3$, where by fast decaying solution we mean a solution which decays faster than the fundamental solution, usually due to cancellation. For general small localized forces, solutions are expected to decay like (1.4). For example, in case of three dimensional exterior domains, it was shown in [14] that leading asymptotic of the solution is a minus one homogeneous profile, which is nothing but one of the Slezkin-Landau solutions of (NS) (see [10]). However, if we assume further certain cancelation of the force, one may expect an extra decay such as (1.6). Indeed, we prove that for such a case the solutions satisfy the decay (1.6) and, in addition, their

asymptotics are given by

$$u(x) = b_0 \nabla E(x) + \sum_{(k,j) \neq (n,n)} a_{jk} \Phi^{jk}(x) + o(|x|^{1-n}) \quad (1.14)$$

for some constants b_0 and a_{jk} , where $\Phi_i^{jk} = \partial_k U_{ij}$ and E is the fundamental solution of Laplace equation (see Proposition 4.6, Theorems 4.8 and 4.9 for the details).

Finally in the Appendix we study the nonexistence of axisymmetric self-similar solutions of (NS) in \mathbb{R}_+^3 under suitable boundary conditions. It is relevant to the asymptotic problem since their existence would be an obstacle to proving (1.11) which has faster decay than self-similar solutions.

In this paper we do not consider the asymptotic formula for two dimensional Navier-Stokes equations, for which we do not know a general existence theory of solutions satisfying the decay (1.6) even in the whole space, because the nonlinear term does not have enough decay. To get existence for dimension two, one usually needs either some symmetry assumptions on the forces (and hence the solutions, see e.g. [6] for aperture problems, [21] for the whole space, and [22] for exterior domains), or the solutions have to be close to some special flows to ensure that the solutions decay sufficiently fast; see e.g. [7].

After a preprint of this paper was posted to arXiv (arXiv:1606.01854v1), Professor D. Iftimie kindly informed us that a formula similar to (1.11) for dimension three, with the asymptotic profile spanned by the Poisson kernel only, also appeared in the thesis of Dr. A. Decaster [3, Remark 4.2.4], with the proof in its Section 4.4. Our error estimate is more refined due to our new Green tensor estimates.

2 Green tensor of the Stokes system in the half space

In this section we derive refined estimates of the Green tensor of the stationary Stokes system in the half space \mathbb{R}_+^n , $n \geq 2$. We first recall in §2.1 the Lorentz tensor, which is the fundamental solution of the stationary Stokes system in \mathbb{R}^n . We then recall in §2.2 the Odqvist tensor, which is the Poisson kernel of the stationary Stokes system in \mathbb{R}_+^n . We finally study in §2.3 the Green tensor.

Let $n \geq 2$ and $E(x)$ and $\Phi(x) = \Phi(|x|)$ be the fundamental solutions of the Laplace and biharmonic equations in \mathbb{R}^n ,

$$-\Delta E = \delta, \quad \Delta^2 \Phi = \delta, \quad (2.1)$$

where δ is the Dirac delta function. Recall

$$E(x) = 2\kappa |x|^{2-n} \quad (n \geq 3); \quad E(x) = -2\kappa \log |x| \quad (n = 2), \quad (2.2)$$

where $\kappa = \frac{1}{2n(n-2)\omega_n}$ if $n \geq 3$ and $\kappa = \frac{1}{4\pi}$ if $n = 2$, $\omega_n = |B_1^{\mathbb{R}^n}| = \frac{2\pi^{n/2}}{n\Gamma(n/2)}$, and $\nabla E = -\frac{x}{n\omega_n|x|^n}$ for all $n \geq 2$. We can integrate $\partial_r(r^{n-1}\Phi') = -r^{n-1}E$ to get an explicit formula for Φ :

$$\begin{aligned} \Phi(x) &= \frac{|x|^2}{8\pi} (\log |x| - 1) \quad (n = 2); & \Phi(x) &= -\kappa \log |x| \quad (n = 4); \\ \Phi(x) &= \frac{\kappa}{(n-4)} |x|^{4-n} \quad (n = 3 \text{ or } n \geq 5). \end{aligned} \quad (2.3)$$

2.1 Lorentz tensor U_{ij} in \mathbb{R}^n

The Lorentz tensor is the fundamental solution of the Stokes system in \mathbb{R}^n , $n \geq 2$, (Lorentz [11], see [16] and [5, §IV.2]). The Lorentz tensor $\vec{U}_j(x) = (U_{ij}(x))_{i=1}^n$ and $q_j(x)$ satisfy, for each fixed $j = 1, \dots, n$,

$$-\Delta \vec{U}_j + \nabla q_j = \delta e_j, \quad \operatorname{div} \vec{U}_j = 0, \quad (x \in \mathbb{R}^n). \quad (2.4)$$

Above e_j is the unit vector in x_j direction. Component-wise,

$$-\Delta U_{ij} + \partial_i q_j = \delta \delta_{ij}, \quad \partial_i U_{ij} = 0, \quad (x \in \mathbb{R}^n). \quad (2.5)$$

Taking div of the first equation of (2.4), we get $\Delta q_j = \partial_j \delta$ in the sense of distributions. In view of (2.1), we can take $q_j = -\partial_j E$. Thus $-\Delta U_{ij} = \delta_{ij} \delta + \partial_i \partial_j E$, and we can take

$$U_{ij} = (-\delta_{ij} \Delta + \partial_i \partial_j) \Phi = \delta_{ij} E + \partial_i \partial_j \Phi, \quad q_j = -\partial_j E. \quad (2.6)$$

For dimension $n = 2$, we have

$$U_{ij}(x) = \frac{1}{4\pi} \left[-\delta_{ij} \log |x| + \frac{x_i x_j}{|x|^2} \right], \quad q_j(x) = \frac{x_j}{2\pi |x|^2}. \quad (2.7)$$

For dimension $n \geq 3$, we have

$$U_{ij}(x) = \frac{1}{2n(n-2)\omega_n} \left[\frac{\delta_{ij}}{|x|^{n-2}} + (n-2) \frac{x_i x_j}{|x|^n} \right], \quad q_j(x) = \frac{x_j}{n\omega_n |x|^n}. \quad (2.8)$$

Summarizing, for $n \geq 2$,

$$U_{ij}(x) = \frac{1}{2} \delta_{ij} E(x) + \frac{1}{2n\omega_n} \frac{x_i x_j}{|x|^n}, \quad q_j(x) = \frac{x_j}{n\omega_n |x|^n}. \quad (2.9)$$

2.2 Odqvist tensor K_{ij} in \mathbb{R}_+^n

The Odqvist tensor K_{ij} is the Poisson kernel for the Stokes system in the half space \mathbb{R}_+^n , $n \geq 2$. A solution (u, p) of the homogeneous Stokes system in the half space \mathbb{R}_+^n with boundary data $\phi : \Sigma = \partial\mathbb{R}_+^n \rightarrow \mathbb{R}^n$ is given by

$$u_i(x) = \int_{\Sigma} K_{ij}(x-z) \phi_j(z) dz, \quad p(x) = \int_{\Sigma} k_j(x-z) \phi_j(z) dz, \quad (2.10)$$

where

$$K_{ij} = 2(\partial_n U_{ij} + \partial_j U_{in} - \delta_{jn} q_i), \quad k_j = -4\partial_j q_n. \quad (2.11)$$

One computes directly using (2.11), (2.9) and (2.7) to get, for $n \geq 2$,

$$K_{ij}(x) = \frac{2x_n x_i x_j}{\omega_n |x|^{n+2}}, \quad k_j(x) = -\partial_j \frac{4x_n}{n\omega_n |x|^n}. \quad (2.12)$$

One can verify that, when $x_n > 0$, using (2.11), $\Delta U_{ij} = \partial_i q_j = -\partial_{ij} E$, and $\partial_i U_{ij} = 0$,

$$\begin{aligned} -\Delta K_{ij} + \partial_i k_j &= 2(-\partial_n \partial_{ij} E - \partial_j \partial_{in} E - 0) + 4\partial_{ij} \partial_n E = 0, \\ \partial_i K_{ij} &= 2(\partial_n \partial_i U_{ij} + \partial_j \partial_i U_{in} + \delta_{jn} \Delta E) = 0. \end{aligned} \quad (2.13)$$

One can also verify that, for $\phi \in C_c^1(\Sigma; \mathbb{R}^n)$,

$$\int_{\Sigma} K_{ij}(x-z) \phi_j(z) dz \rightarrow \phi_i(x') \quad \text{as } x_n \rightarrow 0_+. \quad (2.14)$$

The above is derived by Odqvist [15, §2] using double layer potentials, see also [5, §IV.3]. One may also implicitly derive K_{ij} using Fourier transform in x' as in Solonnikov [17], see also Maekawa-Miura [12].

2.3 Green tensor G_{ij} in \mathbb{R}_+^n

For the Stokes system in the half space \mathbb{R}_+^n , $n \geq 2$, the Green tensor $\vec{G}_j(x, y) = (G_{ij}(x, y))_{i=1}^n$ and $g_j(x, y)$, for each fixed $j = 1, \dots, n$ and $y \in \mathbb{R}_+^n$, satisfy

$$-\Delta_x \vec{G}_j + \nabla_x g_j = \delta_y e_j, \quad \operatorname{div}_x \vec{G}_j = 0, \quad (x \in \mathbb{R}_+^n), \quad (2.15)$$

$$\vec{G}_j(x, y)|_{x_n=0} = 0. \quad (2.16)$$

In components,

$$-\Delta_x G_{ij} + \partial_{x_i} g_j = \delta(x - y) \delta_{ij}, \quad \partial_{x_i} G_{ij} = 0, \quad (x \in \mathbb{R}_+^n), \quad (2.17)$$

$$G_{ij}(x, y)|_{x_n=0} = 0. \quad (2.18)$$

Denote

$$y^* = (y', -y_n) \quad \text{if } y = (y', y_n); \quad \epsilon_j = 1 - 2\delta_{nj}. \quad (2.19)$$

Thus $y_j^* = \epsilon_j y_j$. By (2.9), if $i = j$, then U_{ii} is even in all x_k . If $i \neq j$, U_{ij} is odd in x_i and x_j , but even in x_k if $k \neq i, j$. In particular, with $k = n$,

$$U_{ij}(x^*) = \epsilon_i \epsilon_j U_{ij}(x). \quad (2.20)$$

Let

$$\tilde{G}_{ij}(x, y) = U_{ij}(x - y) - \epsilon_j U_{ij}(x - y^*). \quad (2.21)$$

At $x_n = 0$, with $z = (x', 0) - y$,

$$\tilde{G}_{ij}(x, y)|_{x_n=0} = U_{ij}(z) - \epsilon_j U_{ij}(z^*) = (1 - \epsilon_i) U_{ij}(z) \quad (2.22)$$

by (2.20). Thus

$$\tilde{G}_{ij}(x, y)|_{x_n=0} = 0 \quad (i < n); \quad \tilde{G}_{nj}(x, y)|_{x_n=0} = 2U_{nj}(x' - y', -y_n). \quad (2.23)$$

We can now decompose

$$G_{ij} = \tilde{G}_{ij} + W_{ij}, \quad (2.24)$$

where W_{ij} is given by the boundary layer integral

$$W_{ij}(x, y) = -2 \int_{\Sigma} K_{in}(x - \xi) U_{nj}(\xi - y) d\xi. \quad (2.25)$$

Lemma 2.1. Fix $n \geq 3$.

- (i) $G_{ij}(x, y) = G_{ji}(y, x)$;
- (ii) $G_{ij}(x, y) = \lambda^{n-2} G_{ij}(\lambda x, \lambda y)$ for any $\lambda > 0$.

Proof. (i) The three dimensional case can be found in Odqvist [15, p. 358]. The higher dimensional case is similar: For $x, y \in \mathbb{R}_+^n$, let $\Omega_\epsilon = \mathbb{R}_+^n \setminus (B_\epsilon(x) \cup B_\epsilon(y))$ for $0 < \epsilon \ll 1$. Put the second argument as superscript, e.g., $G(z, x) = G^x(z)$. One has

$$\begin{aligned} 0 &= \lim_{\epsilon \rightarrow 0+} \sum_k \int_{\Omega_\epsilon} \left\{ G_{ki}^x [\Delta_z G_{kj}^y - \partial_{z_k} g_j^y] - G_{kj}^y [\Delta_z G_{ki}^x - \partial_{z_k} g_i^x] \right\} dz \\ &= \lim_{\epsilon \rightarrow 0+} \sum_k \int_{\partial B_\epsilon(x) \cup \partial B_\epsilon(y)} \left\{ G_{ki}^x [\nabla_z G_{kj}^y - e_k g_j^y] - G_{kj}^y [\nabla_z G_{ki}^x - e_k g_i^x] \right\} \cdot \nu \\ &= [0 - G_{ji}(y, x)] - [-G_{ij}(x, y) + 0]. \end{aligned} \quad (2.26)$$

We have used the cancellation of $\nabla_z G_{ki}^x \cdot \nabla_z G_{kj}^y$. We have also used (2.17), (2.18) and the decay at infinity of G_{ij} .

- (ii) It follows from (2.24), (2.25), and the scaling properties of U_{ij} and K_{ij} . \square

In the following we derive an explicit formula for G_{ij} , following the approach of Maz'ja, Plamenevskii, and Stupjalis [13, Appendix 1] for $n = 2, 3$. See [5, IV.3] for formulas for higher dimensions. However, our formula is much more compact, and is suitable for estimates.

Theorem 2.2. Fix $n \geq 2$. For $x, y \in \mathbb{R}_+^n$, denote $w = x - y$, $z = x - y^*$, $\theta = \frac{x_n y_n}{|z|^2} \in (0, \frac{1}{4}]$,

$$Q_s = \frac{1}{|w|^s} - \frac{1}{|z|^s} - \frac{2s x_n y_n}{|z|^{s+2}}, \quad (s > 0); \quad Q_0 = -\log |w| + \log |z| - \frac{2x_n y_n}{|z|^2}. \quad (2.27)$$

Then

$$G_{ij}(x, y) = \delta_{ij} \kappa Q_{n-2} + \frac{1}{2n\omega_n} w_i w_j Q_n + \frac{x_n y_n (w_i w_j + z_i \epsilon_j z_j)}{\omega_n |z|^{n+2}}. \quad (2.28)$$

Remark. Recall $\kappa = \frac{1}{2n(n-2)\omega_n}$ if $n \geq 3$ and $\kappa = \frac{1}{4\pi}$ if $n = 2$. Since

$$|w|^2 = |x' - y'|^2 + x_n^2 + y_n^2 - 2x_n y_n = |z|^2 - 4x_n y_n = |z|^2(1 - 4\theta), \quad (2.29)$$

Q_s is the remainder of the first order Taylor expansion of $|w|^{-s} = |z|^{-s}(1 - 4\theta)^{-s/2}$ when $\theta \ll 1$ for $s > 0$, and similarly for Q_0 as $\log |w| - \log |z| = \frac{1}{2} \log(1 - 4\theta)$. We need Q_0 only if $n = 2$. The definition of Q_s is not continuous in s as $s \rightarrow 0_+$. In fact, $\frac{1}{s} Q_s \rightarrow Q_0$ as $s \rightarrow 0_+$. This discrepancy is related to the choices of the coefficient κ for $n = 2$ and $n \geq 3$.

Proof. Recall $G_{ij} = \tilde{G}_{ij} + W_{ij}$. By (2.9) we may rewrite

$$\begin{aligned} \tilde{G}_{ij}(x, y) &= \frac{1}{2} \delta_{ij} (E(w) - \epsilon_j E(z)) + \frac{1}{2n\omega_n} \left(\frac{w_i w_j}{|w|^n} - \frac{\epsilon_j z_i z_j}{|z|^n} \right) \\ &= \frac{1}{2} \delta_{ij} \left(2\kappa Q_{n-2} + 2\delta_{jn} E(z) + \frac{2x_n y_n}{n\omega_n |z|^n} \right) \\ &\quad + \frac{1}{2n\omega_n} \left(w_i w_j Q_n + \frac{w_i w_j - \epsilon_j z_i z_j}{|z|^n} + \frac{2n x_n y_n w_i w_j}{|z|^{n+2}} \right). \end{aligned} \quad (2.30)$$

Above we have used $2\kappa \cdot 2(n-2) = \frac{2}{n\omega_n}$ for $n \geq 3$ and $2\kappa \cdot 2 = \frac{2}{n\omega_n}$ for $n = 2$.

To compute W_{ij} defined by (2.25), we will use the identity that, for $x, y \in \mathbb{R}_+^n$, $n \geq 2$,

$$\int_{\Sigma} P(x - \xi) E(\xi - y) d\xi = E(x - y^*), \quad P(x) = \frac{2x_n}{n\omega_n |x|^n}. \quad (2.31)$$

It is because $P(x)$ is the Poisson kernel of the Laplace equation in \mathbb{R}_+^n , while $E(x - y^*)$ is the unique bounded (or sublinear if $n = 2$) harmonic function in \mathbb{R}_+^n with the boundary value $E(x - y)|_{x_n=0}$ for fixed y . Note

$$K_{in}(x - \xi) = \frac{2x_n^2(x_i - \xi_i)}{\omega_n |x - \xi|^{n+2}} = \left[\delta_{in} - x_n \frac{\partial}{\partial x_i} \right] P(x - \xi), \quad (2.32)$$

and

$$U_{nj}(\xi - y) = \frac{1}{2} \left[\delta_{nj} - y_n \frac{\partial}{\partial y_j} \right] E(\xi - y). \quad (2.33)$$

Thus, using (2.25) and (2.31),

$$W_{ij}(x, y) = - \left(\delta_{in} - x_n \frac{\partial}{\partial x_i} \right) \left(\delta_{nj} - y_n \frac{\partial}{\partial y_j} \right) \int_{\Sigma} P(x - \xi) E(\xi - y) d\xi \quad (2.34)$$

$$= - \left(\delta_{in} - x_n \frac{\partial}{\partial x_i} \right) \left(\delta_{nj} - y_n \frac{\partial}{\partial y_j} \right) E(x - y^*). \quad (2.35)$$

Expanding the derivatives, with $z = x - y^*$,

$$\begin{aligned} W_{ij}(x, y) = & -\delta_{in}\delta_{jn}E(z) - \frac{\delta_{in}y_n(-\epsilon_j z_j)}{n\omega_n|z|^n} - \frac{\delta_{jn}x_n z_i}{n\omega_n|z|^n} \\ & + \frac{1}{n\omega_n}x_n y_n \left(-\frac{\delta_{ij}\epsilon_j}{|z|^n} + n\frac{z_i\epsilon_j z_j}{|z|^{n+2}} \right). \end{aligned} \quad (2.36)$$

Summing (2.30) and (2.36), and cancelling $\delta_{jn}\delta_{ij}E(z)$, we get

$$G_{ij}(x, y) = \delta_{ij}\kappa Q_{n-2} + \frac{1}{2n\omega_n}w_i w_j Q_n + \frac{x_n y_n (w_i w_j + z_i \epsilon_j z_j)}{\omega_n |z|^{n+2}} + \frac{R}{2n\omega_n |z|^n} \quad (2.37)$$

with

$$R = 2\delta_{ij}x_n y_n + (w_i w_j - \epsilon_j z_i z_j) + 2\delta_{in}y_n \epsilon_j z_j - 2\delta_{jn}x_n z_i - 2\delta_{ij}\epsilon_j x_n y_n = 0. \quad (2.38)$$

The above shows (2.28). \square

We next estimate G_{ij} . For this purpose, it is useful to know the geometry of the level sets of $\theta = \frac{x_n y_n}{|z|^2} \in (0, \frac{1}{4}]$. For fixed $y \in \mathbb{R}_+^n$ and $c \in (0, \frac{1}{4})$, the region $\theta \geq c$ corresponds to a closed disk

$$D_c = \left\{ (x', x_n) : |x' - y'|^2 + (x_n - (\frac{1}{2c} - 1)y_n)^2 \leq \frac{1 - 4c}{4c^2} y_n^2 \right\}, \quad (2.39)$$

which is inside \mathbb{R}_+^n , increases as c decreases, $\cap_{0 < c < 1/4} D_c = D_{1/4} = \{y\}$ and $\cup_{0 < c < 1/4} D_c = \mathbb{R}_+^n$. We also have

$$C^{-1}y_n < |z| < Cy_n \quad \text{if} \quad \frac{1}{10} \leq \theta \leq \frac{1}{4}, \quad (2.40)$$

$$|w| < |z| < C|w| \quad \text{if} \quad 0 < \theta \leq \frac{1}{10}, \quad (2.41)$$

for some constant C independent of $x, y \in \mathbb{R}_+^n$. Estimate (2.40) is because that the radius of D_c is $C(c)y_n$, while (2.41) follows from (2.29). For different $y \in \mathbb{R}_+^n$, their corresponding D_c are translation and dilation of each other.

Lemma 2.3. Fix $n \geq 2$. For $x, y \in \mathbb{R}_+^n$, denote $w = x - y$, $z = x - y^*$, $\theta = \frac{x_n y_n}{|z|^2} \in (0, \frac{1}{4}]$, Q_s be as in (2.27), and

$$R_s = \frac{1}{|w|^s} - \frac{1}{|z|^s}, \quad (s > 0); \quad R_0 = -\log |w| + \log |z|. \quad (2.42)$$

For $x \neq y$ we have, for $s \geq 0$,

$$\begin{aligned} 0 < R_s & \leq C_s |w|^{-s} \theta + C_0 1_{s=0} \log(2 + \frac{y_n}{|w|}), \\ |Q_s| & \leq C_s |w|^{-s} \theta^2 + C_0 1_{s=0} \log(2 + \frac{y_n}{|w|}). \end{aligned} \quad (2.43)$$

Above C_s is independent of $x, y \in \mathbb{R}_+^n$. Moreover, for any $s \geq 0$, for any homogeneous polynomial $g(w')$ of degree $\deg g \geq 0$, for any multi-indices α, β with $\alpha_n = \beta_n = 0$ and $m = |\alpha| + |\beta| > 0$,

$$\nabla_x^\alpha \nabla_y^\beta [g(w') R_s] = \sum_{k=0}^m f_k(w') R_{s+2k}, \quad (2.44)$$

$$\nabla_x^\alpha \nabla_y^\beta [g(w') Q_s] = \sum_{k=0}^m f_k(w') Q_{s+2k}, \quad (2.45)$$

for some homogeneous polynomials $f_k(w')$ with $\deg f_k = \deg g + 2k - m$.

Above and hereafter, the characteristic function 1_ω for a condition ω is 1 if ω is true, and 0 if ω is false. We agree that $f = 0$ if it is a polynomial with negative degree. Note that f_k in (2.44) and (2.45) are the same.

Proof. We first show (2.43). When $\theta > \frac{1}{10}$, we have $|w| < c|z|$ for some $c > 1$ independent of x, y . Thus (2.43) is trivial if $s > 0$, and it is true when $s = 0$ because R_0 and Q_0 are bounded by $1 + \log \frac{|z|}{|w|}$, and by using (2.40).

Suppose now $0 < \theta < \frac{1}{10}$. Recall $|w|^2 = |z|^2(1 - 4\theta)$ by (2.29) and $|w| \sim |z|$. By Taylor expansion,

$$|w|^{-s} = |z|^{-s}(1 - 4\theta)^{-s/2} = |z|^{-s}(1 + 2s\theta + O(\theta^2)), \quad (2.46)$$

$$-\log |w| + \log |z| = -\frac{1}{2} \log(1 - 4\theta) = \frac{1}{2}(4\theta + O(\theta^2)). \quad (2.47)$$

Thus (2.43) follows.

Eqn. (2.44) and (2.45) can be shown by induction on m , using for $j < n$ that

$$\begin{aligned} \partial_{x_j} R_s &= -\partial_{y_j} R_s = -d_s R_{s+2} w_j, \\ \partial_{x_j} Q_s &= -\partial_{y_j} Q_s = -d_s Q_{s+2} w_j, \end{aligned} \quad (2.48)$$

where $d_s = s$ for $s > 0$ and $d_0 = 1$. □

Theorem 2.4. Fix $n \geq 2$. Let $x, y \in \mathbb{R}_+^n$ and $i, j \in \{1, \dots, n\}$. Then

$$|G_{ij}(x, y)| \leq \frac{C_0 x_n y_n}{|x - y|^{n-2} \cdot |x - y^*|^2} + C_0 1_{n=2} \log(2 + \frac{y_n}{|x - y|}). \quad (2.49)$$

Moreover, when $j = n$,

$$|G_{in}(x, y)| \leq \frac{C_0 x_n y_n^2}{|x - y|^{n-2} \cdot |x - y^*|^3} + C_0 1_{n=2} \log(2 + \frac{y_n}{|x - y|}). \quad (2.50)$$

Above C_0 is independent of $x, y \in \mathbb{R}_+^n$.

Proof. Denote $w = x - y$, $z = x - y^*$, and $\theta = \frac{x_n y_n}{|z|^2} \in [0, \frac{1}{4}]$. By Theorem 2.2 and Lemma 2.3, we have

$$|G_{ij}(x, y)| \lesssim |w|^{2-n} \theta^2 + 1_{n=2} \log(2 + \frac{y_n}{|w|}) + \frac{\theta}{|z|^n} |w_i w_j + z_i \epsilon_j z_j|, \quad (2.51)$$

which gives (2.49). In the case $j = n$,

$$\begin{aligned} w_i w_j + \epsilon_j z_i z_j &= w_i(x_n - y_n) - z_i(x_n + y_n) \\ &= (w_i - z_i)x_n - (w_i + z_i)y_n = -\delta_{in} 2y_n x_n - (w_i + z_i)y_n, \end{aligned} \quad (2.52)$$

which is bounded by $|z|y_n$. By this refined estimate and (2.51) we get (2.50). □

Remark. To prove only (2.49) without (2.50), it suffices to use $|w|^2 = |z|^2(1 + O(\theta))$ instead of (2.46) in the proof of Theorem 2.2, and we do not need (2.28).

We next estimate derivatives of $G_{ij}(x, y)$.

Theorem 2.5. Fix $n \geq 2$. Let $x, y \in \mathbb{R}_+^n$ and $i, j \in \{1, \dots, n\}$. Let α and β be multi-indices with $|\alpha| + |\beta| = m > 0$. Then

$$\left| \nabla_x^\alpha \nabla_y^\beta G_{ij}(x, y) \right| \leq \frac{C_m}{|x - y|^{n-2+m}}. \quad (2.53)$$

If $\alpha_n = \beta_n = 0$, we have

$$\left| \nabla_x^\alpha \nabla_y^\beta G_{ij}(x, y) \right| \leq \frac{C_m x_n y_n}{|x - y|^{n-2+m} |z|^2}, \quad (2.54)$$

$$\left| \nabla_x^\alpha \nabla_y^\beta G_{in}(x, y) \right| \leq \frac{C_m x_n y_n^2}{|x - y|^{n-2+m} |z|^3}. \quad (2.55)$$

If $\alpha_n = 0$, we have

$$\left| \nabla_x^\alpha \nabla_y^\beta G_{ij}(x, y) \right| \leq \frac{C_m x_n}{|x - y|^{n-2+m} |z|}. \quad (2.56)$$

Above C_m are independent of $x, y \in \mathbb{R}_+^n$.

Proof. Estimate (2.53) is well-known (see e.g. [5, §IV.3]), and follows from direct differentiation of (2.28), no matter whether $n > 2$ or $n = 2$.

Suppose now $\alpha_n = \beta_n = 0$. By (2.28),

$$\begin{aligned} \nabla_x^\alpha \nabla_y^\beta G_{ij}(x, y) &= \nabla_x^\alpha \nabla_y^\beta \left(\delta_{ij} \kappa Q_{n-2} + \frac{1}{2n\omega_n} w_i w_j Q_n \right) \\ &\quad + x_n y_n \nabla_x^\alpha \nabla_y^\beta \frac{(w_i w_j + z_i \epsilon_j z_j)}{\omega_n |z|^{n+2}} =: \text{I} + \text{II}. \end{aligned} \quad (2.57)$$

By Lemma 2.3,

$$\text{I} = \sum_{k=0}^m \left(f_{ij,k}(z') Q_{n-2+2k} + \tilde{f}_{ij,k}(z') Q_{n+2k} \right), \quad (2.58)$$

for some homogeneous polynomials $f_{ij,k}(z')$ and $\tilde{f}_{ij,k}(z')$ with $\deg f_{ij,k} = 2k - m$ and $\deg \tilde{f}_{ij,k} = 2 + 2k - m$. In particular $f_{ij,0} = 0$. Thus, by Q_s estimates in (2.43) with $s > 0$,

$$|\text{I}| \lesssim \sum_{k=0}^m \left(|w|^{2k-m-(n-2+2k)} \theta^2 + |w|^{2+2k-m-(n+2k)} \theta^2 \right) \lesssim |w|^{2-m-n} \theta^2. \quad (2.59)$$

For II, using $w_i = z_i - 2\delta_{in} y_n$, we may rewrite

$$\begin{aligned} w_i w_j + z_i \epsilon_j z_j &= (z_i - 2\delta_{in} y_n)(z_j - 2\delta_{jn} y_n) + z_i \epsilon_j z_j \\ &= (1 + \epsilon_j) z_i z_j - 2(\delta_{jn} z_i + \delta_{in} z_j) y_n + 4\delta_{jn} \delta_{in} y_n^2. \end{aligned} \quad (2.60)$$

Hence

$$\text{II} = x_n y_n \nabla_x^\alpha \nabla_y^\beta \frac{(1 + \epsilon_j) z_i z_j}{\omega_n |z|^{n+2}} - 2x_n y_n^2 \nabla_x^\alpha \nabla_y^\beta \frac{\delta_{jn} z_i + \delta_{in} z_j}{\omega_n |z|^{n+2}} + x_n y_n^3 \nabla_x^\alpha \nabla_y^\beta \frac{4\delta_{jn} \delta_{in}}{\omega_n |z|^{n+2}}. \quad (2.61)$$

The factors under differentiation are homogeneous rational functions of z of degrees $-n$, $-n-1$, and $-n-2$, respectively. After differentiation they become homogeneous rational functions of z of degrees $-n-m$, $-n-1-m$, and $-n-2-m$, respectively. Thus

$$|\text{II}| \lesssim (1 + \epsilon_j) x_n y_n |z|^{-n-m} + x_n y_n^2 |z|^{-n-1-m} + x_n y_n^3 |z|^{-n-2-m}. \quad (2.62)$$

Summing (2.59) and (2.62) and noting $(1 + \epsilon_j) = 0$ if $j = n$, we get both (2.54) and (2.55). It remains to show (2.56). Using above computations, we note that

$$\begin{aligned} \partial_{y_n}^{\beta_n} \nabla_x^\alpha \nabla_{y'}^{\beta'} G_{ij}(x, y) &= \partial_{y_n}^{\beta_n} \sum_{k=0}^{m-\beta_n} \left(f_{ij,k}(z') Q_{n-2+2k} + \tilde{f}_{ij,k}(z') Q_{n+2k} \right) \\ &\quad + \partial_{y_n}^{\beta_n} \left(x_n y_n \nabla_x^\alpha \nabla_{y'}^{\beta'} \frac{(w_i w_j + z_i \epsilon_j z_j)}{\omega_n |z|^{n+2}} \right) =: J_1 + J_2. \end{aligned} \quad (2.63)$$

Since J_2 is nonsingular and has a factor x_n , it is rather straightforward to obtain (2.56), and thus it suffices to treat J_1 only. In addition, since $f_{ij,k}(z')$ and $\tilde{f}_{ij,k}(z')$ are independent of y_n -variable, we need to estimate only $\partial_{y_n}^{\beta_n} Q_s$ for either $s = n - 2 + 2k$ or $s = n + 2k$. Recalling that $Q_s = R_s - \frac{2sx_n y_n}{|z|^{s+2}}$, it is enough to compute $\partial_{y_n}^{\beta_n} R_s$, since the other term can be treated as J_2 . We will show via induction argument that, for $s > 0$,

$$|\partial_{y_n}^{\beta_n} R_s| \lesssim \frac{x_n}{|w|^{s+\beta_n} |z|}, \quad \beta_n = 0, 1, \dots \quad (2.64)$$

The case $\beta_n = 0$ follows from (2.43). Note

$$\begin{aligned} \partial_{y_n} R_s &= s(x_n - y_n)|w|^{-s-2} + s(x_n + y_n)|z|^{-s-2} \\ &= s(x_n - y_n)R_{s+2} + 2sx_n|z|^{-s-2}. \end{aligned} \quad (2.65)$$

Assume that (2.64) is valid up to $\beta_n = k \geq 0$, and consider $\beta_n = k + 1$:

$$\begin{aligned} \partial_{y_n}^{k+1} R_s &= \partial_{y_n}^k [s(x_n - y_n)R_{s+2} + 2sx_n|z|^{-s-2}] \\ &= s(x_n - y_n)\partial_{y_n}^k R_{s+2} - ks\partial_{y_n}^{k-1} R_{s+2} + 2sx_n\partial_{y_n}^k |z|^{-s-2}. \end{aligned} \quad (2.66)$$

Hence, by induction assumption,

$$|\partial_{y_n}^{k+1} R_s| \lesssim \frac{x_n(x_n - y_n)}{|w|^{s+2+k} |z|} + \frac{x_n}{|w|^{s+k+1} |z|} + \frac{x_n}{|z|^{s+2+k}} \lesssim \frac{x_n}{|w|^{s+1+k} |z|}. \quad (2.67)$$

We can now estimate J_1 using that $|f_{ij,k}| \lesssim |z'|^{2k-m}$ and $|\tilde{f}_{ij,k}| \lesssim |z'|^{2+2k-m}$,

$$|J_1| \lesssim \sum_{k=0}^{m-\beta_n} \left(\frac{|z'|^{2k-m} x_n}{|w|^{n-2+2k+\beta_n} |z|} + \frac{|z'|^{2+2k-m} x_n}{|w|^{n+2k+\beta_n} |z|} \right) \lesssim \frac{x_n}{|w|^{n-2+m+\beta_n} |z|}. \quad (2.68)$$

This completes the proof. \square

Remark 2.6. When $\alpha_n \neq 0$ or $\beta_n \neq 0$, we do not expect (2.54) since ∂_{x_n} or ∂_{y_n} may kill a factor of x_n or y_n . For example, consider the Green function for the Laplace equation in \mathbb{R}_+^3 ,

$$G(x, y) = \frac{1}{4\pi|x-y|} - \frac{1}{4\pi|x-y^*|}. \quad (2.69)$$

We have

$$4\pi\partial_{x_3} G(x, y) = x_3 \left(-\frac{1}{|x-y|^3} + \frac{1}{|x-y^*|^3} \right) + y_3 \left(\frac{1}{|x-y|^3} + \frac{1}{|x-y^*|^3} \right). \quad (2.70)$$

When $y = e_3$ and $1 \leq x_3 \ll |x|$, the first term on the right side is of order $\frac{x_3^2}{|x|^5}$ but the second term is of order $\frac{1}{|x|^3}$. Thus $|\partial_{x_3} G(x, e_3)| \not\lesssim \frac{x_3}{|x|^4}$. However, (2.54) may be still valid if $k < n$:

$$4\pi\partial_{x_1} G(x, y) = (x_1 - y_1) \left(-\frac{1}{|x-y|^3} + \frac{1}{|x-y^*|^3} \right), \quad (2.71)$$

which is $O\left(\frac{x_3}{|x|^4}\right)$ if $y_3 = 1 \leq x_3$.

The following lemma will be used in the proof of Theorem 3.4.

Lemma 2.7. For $n \geq 2$, we have

$$D_{y_l} G_{ij}(x, 0) = \begin{cases} \frac{2x_n x_i x_j}{\omega_n |x|^{n+2}} = K_{ij}(x), & \text{if } j < n = l, \\ 0, & \text{otherwise.} \end{cases} \quad (2.72)$$

Proof. Recall formula (2.28) for $G_{ij}(x, y)$ in Theorem 2.2. Note that $Q_s|_{y_n=0} = 0$ for all $s \geq 0$. By (2.45) of Lemma 2.3, we get

$$\nabla_y^\beta Q_s|_{y_n=0} = 0, \quad \forall s \geq 0, \forall \beta. \quad (2.73)$$

Therefore, when one computes $D_{y_l} G_{ij}(x, 0)$ using (2.28), the first two terms have no contribution and

$$D_{y_l} G_{ij}(x, 0) = 0 + 0 + \frac{x_n \delta_{ln} (1 + \epsilon_j) x_i x_j}{\omega_n |x|^{n+2}}, \quad (2.74)$$

which shows the lemma. \square

Remark. It seems interesting to show the lemma by definitions, not using formula (2.28). Compare the derivation of the Poisson kernel for the Laplace equation from its Green function.

3 Asymptotics of flows in the half space

In this section, we study the spatial asymptotics of stationary solutions of the incompressible Stokes and Navier-Stokes equations in the half space \mathbb{R}_+^n .

We first consider the Stokes system in the half-space,

$$-\Delta v + \nabla p = f + \nabla \cdot F, \quad \operatorname{div} v = 0 \quad \text{in } \mathbb{R}_+^n, \quad (3.1)$$

$$v = 0 \quad \text{on } \partial\mathbb{R}_+^n = \{x_n = 0\}. \quad (3.2)$$

Above $(\nabla \cdot F)_i = \partial_j F_{ji}$. A *weak solution* of (3.1)-(3.2) is a vector field $v \in W_{loc}^{1,2}(\overline{\mathbb{R}_+^n})$ that satisfied the weak form of (3.1) with divergence-free test functions, and (3.2) in the sense of trace, with no assumption on its global integrability in this section.

The following uniqueness result can be found in e.g. [8, Corollary 3.7].

Lemma 3.1 (Uniqueness in \mathbb{R}_+^n). *Let $v \in W_{loc}^{1,2}(\overline{\mathbb{R}_+^n})$, $n \geq 2$, be a weak solution of the Stokes system (3.1)-(3.2) with zero force. If $v(x) = o(|x|)$ as $|x| \rightarrow \infty$, then $v \equiv 0$.*

The following two lemmas show that we can absorb f into $\nabla \cdot F$.

Lemma 3.2 ([9] Lemma 2.5). *If $f(x)$ is defined in \mathbb{R}^n with $|f(x)| \lesssim \langle x \rangle^{-a}$, $a > n \geq 1$, then for any $R > 0$ we can rewrite*

$$f(x) = f_0(x) + \sum_{j=1}^n \partial_j F_j(x) \quad (3.3)$$

where $\operatorname{supp} f_0 \subset B_R(0)$ and $|F_j(x)| \lesssim \langle x \rangle^{-a+1} \|\langle x \rangle^a f(x)\|_{L^\infty}$.

If we are concerned with the half space, the term f_0 can be removed.

Lemma 3.3. *If $g(x)$ is defined in \mathbb{R}_+^n with $|g(x)| \lesssim \langle x \rangle^{-a}$, $a > n \geq 1$, then we can rewrite*

$$g(x) = \sum_{j=1}^n \partial_j G_j(x), \quad (x \in \mathbb{R}_+^n), \quad (3.4)$$

where $|G_j(x)| \lesssim \langle x \rangle^{-a+1} \|\langle x \rangle^a g(x)\|_{L^\infty}$.

Proof. Let

$$f(x) = \begin{cases} g(x - e_n), & (x_n > 1); \\ 0, & (x_n \leq 1). \end{cases} \quad (3.5)$$

By Lemma 3.2, we can decompose $f(x)$ as in (3.3) with $\text{supp } f_0 \in B_{1/2}(0)$. Let $G_j(x) = F_j(x + e_n)$ and we get (3.4) with the desired decay estimate. \square

If the external forces f and F decay sufficiently fast, then bounded solutions of (3.1)-(3.2) have spatial asymptotics of order $-n + 1$. To be more precise, we have the following:

Theorem 3.4 (Asymptotics of Stokes system). *Let $n \geq 2$ and $a > n + 1$. Suppose that v is a weak solution of the Stokes system (3.1)-(3.2) in \mathbb{R}_+^n with $|v(x)| \leq o(|x|)$ as $|x| \rightarrow \infty$. Assume further that $|f(x)| \lesssim \langle x \rangle^{-a}$ and $|F(x)| \lesssim \langle x \rangle^{-a+1}$. Then, $|v(x)| \lesssim \frac{x_n}{\langle x \rangle^n} \lesssim \langle x \rangle^{-n+1}$ and for sufficiently large x ,*

$$v_i(x) = \sum_{j=1}^n K_{ij}(x) b_j + O(\delta(x)), \quad i = 1, \dots, n, \quad (3.6)$$

where

$$b_n = 0, \quad b_j = \int_{\mathbb{R}_+^n} (y_n f_j(y) - F_{nj}(y)) dy, \quad (j < n), \quad (3.7)$$

$$\delta(x) = \frac{x_n}{\langle x \rangle^{\min(n+1, a-1)}} (1 + 1_{a=n+2} \log \langle x \rangle), \quad (3.8)$$

and $K_{ij}(x) = \frac{2x_n x_i x_j}{\omega_n |x|^{n+2}}$ is the Poisson kernel for the Stokes system in the half space.

Remark. Note $\delta(x) = o(\frac{x_n}{\langle x \rangle^n})$ as $|x| \rightarrow \infty$. The asymptotic in (3.6) is spanned by the $n - 1$ vectors $\{\vec{K}_j : 1 \leq j \leq n - 1\}$, where $(\vec{K}_j)_i = K_{ij}$. That \vec{K}_n is not present is because a solution of (3.1)-(3.2) should have zero flux on any hemisphere $S_R^+ = \{x \in \mathbb{R}_+^n, |x| = R\}$, while \vec{K}_n has nonzero flux. Note that the flux of the error term of (3.6) on S_R^+ vanishes as $R \rightarrow \infty$.

Proof. By Lemma 3.3, we may write $f_j = \partial_i \tilde{F}_{ij}$ where $|\tilde{F}_{ij}(x)| \lesssim \langle x \rangle^{-a-1}$. We have

$$\int_{\mathbb{R}_+^n} y_n f_j(y) dy = \int_{\mathbb{R}_+^n} y_n \partial_i \tilde{F}_{ij}(y) dy = - \int_{\mathbb{R}_+^n} \tilde{F}_{nj}. \quad (3.9)$$

By absorbing \tilde{F} into F , we may assume $f = 0$.

By uniqueness Lemma 3.1, we have the representation formula,

$$v_i(x) = - \sum_{j=1}^n \sum_{\alpha=1}^n \int_{\mathbb{R}_+^n} \partial_{y_\alpha} G_{ij}(x, y) F_{\alpha j}(y) dy = I_1 + I_2, \quad (3.10)$$

where

$$\begin{aligned} I_1 &:= - \sum_{j=1}^n \sum_{\alpha=1}^n \int_{\mathbb{R}_+^n} \partial_{y_\alpha} G_{ij}(x, 0) F_{\alpha j}(y) dy, \\ I_2 &:= - \sum_{j=1}^n \sum_{\alpha=1}^n \int_{\mathbb{R}_+^n} (\partial_{y_\alpha} G_{ij}(x, y) - \partial_{y_\alpha} G_{ij}(x, 0)) F_{\alpha j}(y) dy. \end{aligned} \quad (3.11)$$

We first compute I_1 . By (2.72) in Lemma 2.7, the summand in I_1 is nonzero only if $j < n = \alpha$ and

$$I_1 = - \sum_{j=1}^{n-1} \frac{2x_n x_i x_j}{\omega_n |x|^{n+2}} \int_{\mathbb{R}_+^n} F_{nj}(y) dy = \sum_{j=1}^{n-1} K_{ij}(x) b_j. \quad (3.12)$$

Secondly, we estimate I_2 . We may assume $|x| > 10$. For notational convenience, for given x we denote $A_x = \{y \in \mathbb{R}_+^n : |y| \leq \frac{|x|}{2}\}$ and $B_x = \mathbb{R}_+^n \setminus A_x$.

$$\begin{aligned} I_2 &= - \sum_{j=1}^n \sum_{\alpha=1}^n \int_{\mathbb{R}_+^n} (\partial_{y_\alpha} G_{ij}(x, y) - \partial_{y_\alpha} G_{ij}(x, 0)) F_{\alpha j}(y) dy \\ &= \int_{A_x} \cdots dy + \int_{B_x} \cdots dy := J_1 + J_2. \end{aligned} \quad (3.13)$$

By (2.56) of Theorem 2.5,

$$\begin{aligned} |J_1| &\leq C \int_{|y| \leq \frac{|x|}{2}} \sup_{|\tilde{y}| \leq \frac{|x|}{2}} |\nabla_y^2 G_{ij}(x, \tilde{y})| |y F_{\alpha j}(y)| dy \\ &\leq \frac{C x_n}{|x|^{n+1}} \int_{|y| \leq \frac{|x|}{2}} \langle y \rangle^{2-a} dy \leq C \delta(x). \end{aligned} \quad (3.14)$$

Above we have used that, for $m \geq 0$ and $R > 0$,

$$\int_{|y| \leq R} \langle y \rangle^{-m} dy \lesssim \begin{cases} 1 & \text{if } m > n, \\ \log \langle R \rangle & \text{if } m = n, \\ \langle R \rangle^{n-m} & \text{if } 0 \leq m < n, \end{cases} \approx \langle R \rangle^{(n-m)_+} (1 + 1_{m=n} \log \langle R \rangle), \quad (3.15)$$

with $m = a - 2$. Recall $(r)_+ = \max(r, 0)$. For J_2 ,

$$\begin{aligned} |J_2| &\leq \sum_{j=1}^n \sum_{\alpha=1}^n \int_{\frac{|x|}{2} < |y|} (|\partial_{y_\alpha} G_{ij}(x, y)| + |\partial_{y_\alpha} G_{ij}(x, 0)|) |F_{\alpha j}(y)| dy \\ &\leq C \int_{\frac{|x|}{2} < |y|} \left(\frac{x_n}{|x - y|^{n-1} |x - y^*|} + \frac{x_n}{|x|^n} \right) |y|^{1-a} dy \leq C \frac{x_n}{|x|^{a-1}}. \end{aligned} \quad (3.16)$$

This completes the proof. \square

Remark 3.5. If $a \leq n + 1$, the integral (3.7) for b_j diverges and the asymptotic formula (3.6) is meaningless. However, the integral (3.10) still converges if $1 < a < \infty$ and

$$|v(x)| \lesssim \int_{\mathbb{R}_+^n} \frac{x_n}{|x - y|^{n-1} |x - y^*|} \langle y \rangle^{1-a} dy. \quad (3.17)$$

By estimating the integral in the two regions $\{|y| < |x|/2\}$ and $\{|y| > |x|/2\}$ separately as in (3.13),

$$|v(x)| \lesssim \frac{x_n}{\langle x \rangle^{\min(n, a-1)}} (1 + 1_{a=n+1} \log \langle x \rangle). \quad (3.18)$$

Next we consider the Navier-Stokes equations in the half-space, i.e.,

$$-\Delta u + (u \cdot \nabla) u + \nabla p = f + \nabla \cdot F, \quad \operatorname{div} u = 0 \quad \text{in } \mathbb{R}_+^n, \quad (3.19)$$

$$u = 0 \quad \text{on } \partial \mathbb{R}_+^n = \{x_n = 0\}. \quad (3.20)$$

A *weak solution* u of (3.19)-(3.20) is a weak solution of (3.1)-(3.2) with force $f + \nabla \cdot (F - u \otimes u)$.

If the decay rates of external forces f and F are sufficiently fast with small coefficient, there exist solutions of the Navier-Stokes equations (3.19)-(3.20), whose spatial asymptotics is of $-n + 1$ -order. Our result reads as follows:

Theorem 3.6 (Existence and asymptotics of NSE). *Let $n \geq 3$ and $a > n + 1$. There exists $\epsilon_0 > 0$ such that if $|f(x)| \leq \epsilon \langle x \rangle^{-a}$ and $|F(x)| \leq \epsilon \langle x \rangle^{-a+1}$ with $\epsilon < \epsilon_0$, then there exists a weak solution u of the Navier-Stokes equations (3.19)-(3.20) in \mathbb{R}_+^n with $|u(x)| \lesssim \frac{\epsilon x_n}{\langle x \rangle^n} \lesssim \epsilon \langle x \rangle^{-n+1}$ and, furthermore, its asymptotics is given as*

$$u_i(x) = \sum_{j=1}^n K_{ij}(x) \tilde{b}_j + O(\epsilon \tilde{\delta}(x)), \quad (3.21)$$

where

$$\tilde{b}_n = 0, \quad \tilde{b}_j = \int_{\mathbb{R}_+^n} \{u_n(y)u_j(y) + y_n f_j(y) - F_{nj}(y)\} dy, \quad (j < n), \quad (3.22)$$

$$\tilde{\delta}(x) = \frac{x_n}{\langle x \rangle^{\min(n+1, a-1)}} (1 + 1_{\tilde{a}=n+2} \log \langle x \rangle), \quad \tilde{a} = \min(a, 2n-1), \quad (3.23)$$

and $K_{ij}(x) = \frac{2x_n x_i x_j}{\omega_n |x|^{n+2}}$ is the Poisson kernel for the Stokes system in the half space.

Unlike Theorem 3.4, the case $n = 2$ is not included in Theorem 3.6. Note $2n - 1 \geq n + 2$ and $\tilde{a} > n + 1$ due to $n > 2$.

Proof. As in the proof of Theorem 3.4, we may assume $f = 0$. Let

$$\mathcal{K} = \left\{ v \in C(\overline{\mathbb{R}_+^n}; \mathbb{R}^n) : v|_{\partial \mathbb{R}_+^n} = 0, \|v\|_{\mathcal{K}} := \sup_{x \in \mathbb{R}_+^n} \langle x \rangle^{n-1} |v(x)| < C\epsilon \right\}, \quad (3.24)$$

where $0 < \epsilon \ll 1$ is sufficiently small and will be specified later. Now we set $v^1(x) = 0$ and iteratively define v^{k+1} , $k = 1, 2, \dots$, by

$$v_i^{k+1}(x) = \int_{\mathbb{R}_+^n} (\partial_{y_\alpha} G_{ij})(x, y) (v_\alpha^k v_j^k - F_{\alpha j})(y) dy, \quad (3.25)$$

which solves the Stokes system

$$-\Delta v^{k+1} + \nabla p^{k+1} = -(v^k \cdot \nabla) v^k + \nabla \cdot F, \quad \operatorname{div} v^{k+1} = 0 \quad \text{in } \mathbb{R}_+^n, \quad (3.26)$$

$$v^{k+1} = 0 \quad \text{on } \partial \mathbb{R}_+^n. \quad (3.27)$$

Due to Theorem 3.4, we have $\|v^{k+1}\|_{\mathcal{K}} \leq C\epsilon$ uniformly for all $k = 1, 2, \dots$, if ϵ is sufficiently small. If we set $\delta v^k := v^{k+1} - v^k$, we have

$$\delta v_i^{k+1}(x) = \int_{\mathbb{R}_+^n} (\partial_{y_\alpha} G_{ij})(x, y) (v_\alpha^{k+1} \delta v_j^k + v_j^k \delta v_\alpha^k)(y) dy, \quad (3.28)$$

and hence

$$\|\delta v^{k+1}\|_{\mathcal{K}} \leq C\epsilon \|\delta v^k\|_{\mathcal{K}}. \quad (3.29)$$

The argument of contraction mapping gives a unique solution u of

$$u_i(x) = \int_{\mathbb{R}_+^n} (\partial_{y_\alpha} G_{ij})(x, y) (u_\alpha u_j - F_{\alpha j})(y) dy, \quad \|u\|_{\mathcal{K}} \leq C\epsilon. \quad (3.30)$$

Finally, we may consider u as a solution of the Stokes system with force tensor $F_{\alpha j} - u_\alpha u_j$. Since $|(F_{\alpha j} - u_\alpha u_j)(x)| \leq C\epsilon \langle x \rangle^{-\tilde{a}+1}$ with $\tilde{a} = \min(a, 2n-1)$, Theorem 3.4 gives the desired asymptotics (3.21)-(3.23). \square

Next theorem shows that for any vector $b = (b_1, \dots, b_{n-1}, 0)$ with small magnitude the Navier-Stokes equations (3.19)-(3.20) has a solution whose leading asymptotics is $\sum_{j=1}^{n-1} K_{ij} b_j$. More precisely, we have the following.

Theorem 3.7 (Asymptotic completeness). *Let $n \geq 3$. There exists a small number $\epsilon_1 > 0$ such that if $b = (b_1, \dots, b_{n-1}, 0)$, $|b| = \epsilon < \epsilon_1$, then there exists a smooth 2-tensor F supported in $B_1 \cap \mathbb{R}_+^n$ and a weak solution u of the Navier-Stokes equations (3.19)-(3.20) corresponding to this F and zero f , satisfying*

$$u_i(x) = \sum_{j=1}^{n-1} K_{ij} b_j + O(\tilde{\delta}(x)), \quad (3.31)$$

where $\tilde{\delta}(x)$ is given by (3.23).

Proof. Fix any smooth scalar function ϕ supported in $B_1 \cap \mathbb{R}_+^n$ with $\int \phi = 1$. For small $a = (a_1, \dots, a_{n-1})$, define 2-tensor F^a by

$$F_{ij}^a = 0 \quad \text{if } i < n; \quad F_{ij}^a = -a_j \phi \quad \text{if } i = n. \quad (3.32)$$

By Theorem 3.6, there is a solution u^a of the Navier-Stokes equations with force F^a and zero f if $|a| \leq \epsilon_0$ for some small $\epsilon_0 > 0$. We have $|u^a(x)| \leq C|a|\langle x \rangle^{1-n}$. The coefficients $(\tilde{b}_1, \dots, \tilde{b}_{n-1})$ of the leading term in (3.21) for u^a will be denoted as $B_{NS}(a)$. Thus $B_{NS}(a)_j = a_j + \int_{\mathbb{R}_+^n} u_n^a u_j^a$, and for some C_1 ,

$$|B_{NS}(a)_j - a_j| = \left| \int_{\mathbb{R}_+^n} u_n^a u_j^a \right| \leq \frac{C_1}{n} |a|^2, \quad |B_{NS}(a) - B_{NS}(\tilde{a})| \leq C_1 |a - \tilde{a}|. \quad (3.33)$$

For given small b we want to solve a so that $B_{NS}(a) = b$. This equation can be rewritten as a fixed point problem

$$a = \Phi(a), \quad \Phi(a) := a - B_{NS}(a) + b. \quad (3.34)$$

Denote $D_r = \{a \in \mathbb{R}^{n-1} : |a| \leq r\}$. One checks easily that Φ is continuous on D_{ϵ_0} . Denote $\epsilon_1 = \min(\epsilon_0/2, \frac{1}{4C_1})$. Suppose $|b| = \epsilon \leq \epsilon_1$. For $a \in D_{2\epsilon}$, we have

$$|\Phi(a)| \leq |a - B_{NS}(a)| + |b| \leq C_1(2\epsilon)^2 + \epsilon \leq 2\epsilon. \quad (3.35)$$

Thus Φ is a continuous map that maps the closed disk $D_{2\epsilon}$ into itself. By Brouwer fixed point theorem, Φ has a fixed point in $D_{2\epsilon}$. This completes the proof. \square

The next theorem is an application of Theorem 3.6 and considers the asymptotics of any given solution.

Theorem 3.8 (Asymptotics). *Let $n \geq 3$ and $a > n + 1$. Suppose that $u \in W_{loc}^{1,2}(\overline{\mathbb{R}_+^n})$ is a weak solution of the Navier-Stokes equations (3.19)-(3.20) with force $f + \nabla \cdot F$.*

(i) *Suppose $|u(x)| \leq C\langle x \rangle^{-m}$, $m > \max\{\frac{n-2}{2}, \frac{n-1}{3}, \frac{n}{4}\}$, $|f(x)| \leq \epsilon\langle x \rangle^{-a}$ and $|F(x)| \leq \epsilon\langle x \rangle^{-a+1}$ for sufficiently small ϵ . Then, u agrees with the solution of Theorem 3.6, $|u(x)| \leq C\epsilon x_n \langle x \rangle^{-n}$, and its asymptotics is given by (3.21) with \tilde{b}_j and $\tilde{\delta}(x)$ given by (3.22) and (3.23).*

(ii) *Suppose $|f(x)| \leq C\langle x \rangle^{-a}$ and $|F(x)| \leq C\langle x \rangle^{-a+1}$, and $|u(x)| \leq C\langle x \rangle^{-1-\sigma}$ for some $\sigma > 0$. Then, $|u(x)| \leq Cx_n \langle x \rangle^{-n}$, and its asymptotics is given as*

$$u_i(x) = \sum_{j=1}^n K_{ij}(x) \tilde{b}_j + O(\tilde{\delta}(x)), \quad (3.36)$$

with \tilde{b}_j and $\tilde{\delta}(x)$ given by (3.22) and (3.23).

(iii) Suppose $|f(x)| \leq C\langle x \rangle^{-a}$ and $|F(x)| \leq C\langle x \rangle^{-a+1}$, and $|u(x)| \leq \epsilon\langle x \rangle^{-1}$ for sufficiently small ϵ . Then, $|u(x)| \leq Cx_n\langle x \rangle^{-n}$, and its asymptotics is given by (3.36) with \tilde{b}_j and $\tilde{\delta}(x)$ given by (3.22) and (3.23).

Note that Case (i) assumes small f and F but allows large u , Case (ii) allows large f , F and u but assumes extra decay, and Case (iii) assumes small u but allows large f and F . Also note that we do not claim smallness in Case (iii). The error estimate has a small factor ϵ only in Case (i).

Proof. As in the previous proofs, we assume that $f = 0$ without loss of generality.

• Case (i). We may assume $m < n - 1$. By Theorem 3.6, there exists a solution \tilde{u} , which satisfies the conclusion of Theorem 3.6. Thus, it suffices to show $u = \tilde{u}$. Set $w = u - \tilde{u}$ and $q = p - \tilde{p}$. We get

$$-\Delta w + \nabla q = -(u \cdot \nabla)w - (w \cdot \nabla)\tilde{u}, \quad \operatorname{div} w = 0 \quad \text{in } \mathbb{R}_+^n, \quad (3.37)$$

$$w = 0 \quad \text{on } \partial\mathbb{R}_+^n = \{x_n = 0\}. \quad (3.38)$$

By [8, Theorem 3.4], for $R > 1$,

$$\|\nabla w\|_{L^2(B_R^+)} + \|q - q_R\|_{L^2(B_R^+)} \leq C\|u \otimes w + w \otimes \tilde{u}\|_{L^2(B_{2R}^+)} + \frac{C}{R}\|w\|_{L^2(B_{2R}^+)} \leq C, \quad (3.39)$$

where $q_R = |B_{2R}^+|^{-1} \int_{B_{2R}^+} q$ and C is independent of w and R . The second inequality is due to $|u(x)| + |\tilde{u}(x)| + |w(x)| \lesssim \langle x \rangle^{-m}$. In particular $\nabla w \in L^2(\mathbb{R}_+^n)$.

Let $Z \in C^2(\mathbb{R})$ with $0 \leq Z(t) \leq 1$, $Z(t) = 0$ for $t > 1$ and $Z(t) = 1$ for $t < 1/2$. Let $\zeta = Z(|x|/R)$. Testing (3.37) with $w\zeta^2$ and integrating by parts, we get

$$\int |\nabla(w\zeta)|^2 = \int \left\{ qw_i \partial_i \zeta^2 + \frac{|w|^2}{2} u_i \partial_i \zeta^2 + |w|^2 |\nabla \zeta|^2 + w_j \tilde{u}_i [w_i \zeta \partial_j \zeta + \zeta \partial_j (w_i \zeta)] \right\} = \sum_{j=1}^5 I_j. \quad (3.40)$$

Note $|I_2| + |I_4| \lesssim R^{n-1-3m}$, $|I_3| \lesssim R^{n-2-2m}$, Also,

$$I_1 = \int (q - q_R) w \cdot \nabla \zeta^2 \leq \|q - q_R\|_{L^2(B_R^+)} \|w \cdot \nabla \zeta^2\|_{L^2(B_R^+)} \lesssim R^{\frac{n}{2}-1-m} \quad (3.41)$$

using (3.39). Finally,

$$|I_5| \leq \|\tilde{u}\|_{L^n} \|w\zeta\|_{L^{\frac{2n}{n-2}}} \|\nabla(w\zeta)\|_{L^2} \leq C\epsilon \int |\nabla(w\zeta)|^2. \quad (3.42)$$

If $C\epsilon < 1$, we get $\int |\nabla(w\zeta)|^2 \leq o(1)$. Taking $R \rightarrow \infty$, we get $\nabla w = 0$ and $w = 0$.

• Case (ii). We may assume $0 < \sigma < n - 2$. By uniqueness (Lemma 3.1), we have the representation formula,

$$u_i(x) = \sum_{j=1}^n \sum_{\alpha=1}^n \int_{\mathbb{R}_+^n} \partial_{y_\alpha} G_{ij}(x, y) (u_\alpha u_j - F_{\alpha j})(y) dy. \quad (3.43)$$

The contribution from F is bounded by $\frac{Cx_n}{\langle x \rangle^n}$ by Theorem 3.4. The nonlinearity satisfies $|u_\alpha u_j(y)| \leq C\langle y \rangle^{1-a'}$ with $a' = 2\sigma + 3 > 1$. By Remark 3.5,

$$|u(x)| \lesssim \frac{x_n}{\langle x \rangle^{\min(n, a'-1)}} (1 + 1_{a'=n+1} \log \langle x \rangle) + \frac{x_n}{\langle x \rangle^n}. \quad (3.44)$$

If $a' - 1 \leq n$, we can avoid the log factor by taking a slightly smaller a' and we get

$$|u(x)| \lesssim \frac{x_n}{\langle x \rangle^{\frac{3}{2}\sigma+2}} \lesssim \langle x \rangle^{-1-\frac{3}{2}\sigma}. \quad (3.45)$$

We can repeat this procedure until we obtain $a' - 1 > n$ and hence $|u(x)| \leq \frac{Cx_n}{|x|^n}$. We then use Theorem 3.4 to get its asymptotics.

• Case (iii). Fix $\sigma \in (0, 1)$. We will construct a solution v satisfying $|v(x)| \leq C\langle x \rangle^{-1-\sigma}$ and the following perturbed equations

$$-\Delta v + \nabla \pi + (u \cdot \nabla)v = \nabla F, \quad \operatorname{div} v = 0 \quad \text{in } \mathbb{R}_+^n, \quad (3.46)$$

$$v = 0 \quad \text{on } \partial\mathbb{R}_+^n = \{x_n = 0\}. \quad (3.47)$$

This can be done by iteration: Let $v^{(0)} = 0$ and define v^{k+1} for $k \geq 0$ by

$$v_i^{k+1}(x) = \int_{\mathbb{R}_+^n} (\partial_{y_\alpha} G_{ij})(x, y) (u_\alpha^k v_j^k - F_{\alpha j})(y) dy. \quad (3.48)$$

For ϵ sufficiently small, $|v^{(k+1)}(x)| \leq C\langle x \rangle^{-1-\sigma}$ uniformly in k using Remark 3.5, and converges to some v with the same bound. The difference $w = u - v$ satisfies

$$w_i(x) = \int_{\mathbb{R}_+^n} (\partial_{y_\alpha} G_{ij})(x, y) (u_\alpha^k w_j^k)(y) dy, \quad |w(x)| \leq C\langle x \rangle^{-1-\sigma}. \quad (3.49)$$

By Remark 3.5,

$$\|\langle x \rangle^{1+\sigma} w(x)\|_{L^\infty} \leq C\epsilon \|\langle x \rangle^{1+\sigma} w(x)\|_{L^\infty}. \quad (3.50)$$

Thus, if ϵ is sufficiently small, $w = 0$ and $|u(x)| \leq C\langle x \rangle^{-1-\sigma}$.

By Case (ii), we deduce $|u(x)| \leq Cx_n \langle x \rangle^{-n}$. \square

Another application of Theorem 3.6 is on asymptotic profiles of solutions for the aperture type problem of the Navier-Stokes equations. Let $\Sigma_r := \partial\mathbb{R}_+^n \setminus B_r = \{(x', 0) : |x'| \geq r\}$ and $\Omega_r = \mathbb{R}_+^n \setminus \overline{B}_r$, and consider

$$-\Delta u + (u \cdot \nabla)u + \nabla p = 0, \quad \operatorname{div} u = 0 \quad \text{in } \Omega_r, \quad (3.51)$$

$$u = 0 \quad \text{on } \Sigma_r. \quad (3.52)$$

We emphasize that no boundary condition is imposed on $\partial B_r \cap \mathbb{R}_+^n$.

Suppose that $|u(x)| \lesssim |x|^{-1-\delta}$ for large x and is small for $r \leq |x| \leq \rho < \infty$. Choose $r < l_1 < l_2 < \rho$ and let ζ be a smooth cut-off function satisfying

$$\zeta(x) = \begin{cases} 1 & \text{if } |x| > l_2 \\ 0 & \text{if } |x| < l_1. \end{cases} \quad (3.53)$$

Recall that $\vec{K}_n = (K_{1n}, \dots, K_{nn})$ is the Poisson kernel for the Stokes system in the half space with $j = n$ and k_n is the pressure corresponding to \vec{K}_n . Set

$$w = (u - \tilde{b}_n K_n) \zeta + \tilde{w}, \quad \pi = (p - \tilde{b}_n k_n) \zeta, \quad (3.54)$$

where \tilde{w} solves

$$\operatorname{div} \tilde{w} = (u - \tilde{b}_n K_n) \cdot \nabla \zeta \quad \text{in } B_{l_2}^+ \setminus B_{l_1}, \quad \tilde{w} = 0 \quad \text{on } \partial(B_{l_2}^+ \setminus B_{l_1}). \quad (3.55)$$

Then, w solves

$$-\Delta w + (w \cdot \nabla)w + \nabla \pi = f + \nabla \cdot F, \quad \operatorname{div} w = 0 \quad \text{in } \mathbb{R}_+^n, \quad (3.56)$$

$$w = 0 \quad \text{on } \partial \mathbb{R}_+^n = \{x_n = 0\}, \quad (3.57)$$

where f and $F = (F_{\alpha j})_{\alpha, j=1, \dots, n}$ are given by

$$f = (u - \tilde{b}_n K_n) \Delta \zeta + (p - \tilde{b}_n k_n) \nabla \zeta + u \cdot \nabla \zeta u, \quad (3.58)$$

$$\begin{aligned} F_{\alpha j} = & -2\partial_\alpha \zeta (u - \tilde{b}_n K_n)_j - \delta_{\alpha j} \partial_j \tilde{w} + \tilde{w}_\alpha ((u - \tilde{b}_n K_n)_\zeta + \tilde{w})_j + (u - \tilde{b}_n K_n)_\alpha \zeta \tilde{w}_j \\ & - (\tilde{b}_n K_n)_\alpha \zeta (u - \tilde{b}_n K_n)_j \zeta - u_\alpha \zeta (\tilde{b}_n K_n)_j \zeta - u_\alpha (1 - \zeta) u_j \zeta. \end{aligned} \quad (3.59)$$

Note that f is small and has compact support, while F is small and decays like $|x|^{1-a}$, $a = n + 1 + \delta$. Thus, if $\delta > 0$, the asymptotic profile of u is $\tilde{b}_n K_n$ plus that of w given by (3.22). To be more precise, we have the following.

Theorem 3.9 (Aperture type problem). *Let $n \geq 3$, $0 < r < \rho < \infty$, and $0 < \delta < 1$. There is a small $\epsilon_0 > 0$ such that, if $u \in H_{loc}^1(\overline{\Omega}_r)$ is a weak solution of (3.51) and (3.52) in Ω_r satisfying $|u(x)| \leq \langle x \rangle^{-1-\delta}$ and $\epsilon = \|u\|_{L^\infty(B_\rho \setminus B_r)} \leq \epsilon_0$, then $|u(x)| \lesssim \epsilon \langle x \rangle^{-n+1}$ in Ω_r and its asymptotics is given by*

$$u_i(x) = \sum_{j=1}^n K_{ij} \tilde{b}_j + O\left(\frac{\epsilon x_n}{\langle x \rangle^{n+1}} (1 + 1_{n=3} \log \langle x \rangle)\right), \quad (3.60)$$

where $\tilde{b}_n = \int_{\partial B_r \cap \mathbb{R}_+^n} u \cdot \nu d\sigma$, and \tilde{b}_j for $j < n$ is given in (3.22) with f and F in (3.58) and (3.59).

Proof. Choose $r < l_1 < l_2 < \rho$. Taking a partition of unity for the region $B_\rho^+ \setminus B_r$, and using the pressure-independent interior and boundary estimates in [18] and [8, Theorem 3.8], we get

$$\|\nabla p\|_{L^{n+1}(B_{l_2}^+ \setminus B_{l_1})} \leq C\epsilon. \quad (3.61)$$

Replacing p by $p - \bar{p}$ where \bar{p} is the average of p in $B_{l_2}^+ \setminus B_{l_1}$, we also have $|p| < C\epsilon$ in $B_{l_2}^+ \setminus B_{l_1}$ by Sobolev imbedding.

Recall that the cut-off w defined in (3.54) satisfies the Navier-Stokes system (3.56)–(3.57) in \mathbb{R}_+^n with force $f + \nabla F$ given in (3.58) and (3.59). Note $|\tilde{b}_n| \leq C\epsilon$, both \tilde{w} and f have compact supports, $|\tilde{w}(x)| + |f(x)| \leq C\epsilon$, and $|F(x)| \leq C\epsilon \langle x \rangle^{1-a}$ with $a = n + 1 + \delta$ by the hypothesis. By assumption $|w(x)| \leq C\langle x \rangle^{-1-\sigma}$.

We now first apply Theorem 3.8 (ii) to get $|w(x)| \leq C\langle x \rangle^{1-n}$, which yields the refined decay estimate $|F(x)| \leq C\epsilon \langle x \rangle^{-(2n-2)}$.

We next apply Theorem 3.8 (i) to get $|w(x)| \leq C\epsilon \langle x \rangle^{1-n}$ and the asymptotic formula (3.60). \square

We remark that similar asymptotics as (3.60) are known in [1, Theorem 6.3] for an aperture problem in dimension three (see also [5, Theorem 9.1]). The error term presented in [1] is of $O(\langle x \rangle^{-2-\eta})$ for any $\eta \in (0, 1)$ and the error term in (3.60) is slightly better in the sense of the log correction, as well as the presence of an anisotropic effect, namely $O\left(\frac{\epsilon x_3}{\langle x \rangle^4} (1 + \log \langle x \rangle)\right)$ in three dimensions.

4 Asymptotics of fast decaying flows in the whole space and exterior domains

In this section we study the asymptotic profiles of *fast decaying* Stokes and Navier-Stokes flows in \mathbb{R}^n and exterior domains. It is well-known that the generic decay rate of these flows are $|x|^{-n+2}$. Our concern here is flows with faster decay $|x|^{-n+1}$, usually due to some cancellation of the force.

We first choose a basis. For $j, k = 1, 2, \dots, n$, we define the vector fields

$$\Phi^{jk} = (\Phi_1^{jk}, \dots, \Phi_n^{jk}), \quad \Phi_i^{jk} = \partial_k U_{ij}. \quad (4.1)$$

Obviously, for $n \geq 2$,

$$|\Phi^{jk}(x)| \leq C|x|^{-n+1}, \quad |\nabla \Phi^{jk}(x)| \leq C|x|^{-n}. \quad (4.2)$$

We will show that the asymptotic profile of a fast decaying flow is given by the linear combination of the vector fields Φ^{jk} , $(j, k) \neq (n, n)$. We first collect some properties of Φ^{jk} .

Lemma 4.1. *Let $n \geq 2$. The set*

$$\left\{ \Phi^{jk} : 1 \leq j, k \leq n, \quad (j, k) \neq (n, n) \right\} \quad (4.3)$$

consists of $n^2 - 1$ linearly independent vector fields.

Proof. We first note, by differentiating (2.9),

$$\Phi_i^{jk}(x) = \partial_k U_{ij}(x) = \frac{c_n}{|x|^n} \left[(\delta_{jk} - \frac{nx_j x_k}{|x|^2}) x_i + \delta_{ik} x_j - \delta_{ij} x_k \right]. \quad (4.4)$$

We choose $|x| = 1$ and we may omit c_n in the following argument. Φ^{jk} is written as

$$\Phi^{jk}(x) = \begin{cases} (1 - nx_j^2)x & (k = j), \\ u^{jk} + v^{jk} & (k \neq j), \end{cases} \quad (4.5)$$

$$u_i^{jk} = -nx_j x_k x_i, \quad v_i^{jk} = \delta_{ik} x_j - \delta_{ij} x_k.$$

Note that $u^{jk} = u^{kj} = \frac{1}{2}(\Phi^{jk} + \Phi^{kj})$ and $v^{jk} = -v^{kj} = \frac{1}{2}(\Phi^{jk} - \Phi^{kj})$ for $k \neq j$. Hence

$$\text{span} \{ \Phi^{jk}, \Phi^{kj} \} = \text{span} \{ u^{jk}, v^{jk} \} \quad \forall k \neq j, \quad (4.6)$$

and

$$\text{span}_{k \neq j} \{ \Phi^{jk} \} = \text{span}_{k < j} \{ u^{jk}, v^{jk} \}. \quad (4.7)$$

It is easy to see that the set $\{v^{jk} : k < j\}$ contains $\frac{1}{2}n(n-1)$ linearly independent vectors which are orthogonal to x .

On the other hand, the set

$$\{u^{jk} : k < j\} \cup \{\Phi^{jj} : j < n\} \quad (4.8)$$

contains $\frac{1}{2}n(n-1) + (n-1)$ vectors which are of the form $\phi(x)x$. We claim this set is linearly independent: If

$$f(x) := \sum_{k < j} a_{jk} x_j x_k + \sum_{l < n} b_l (1 - nx_l^2) = 0, \quad (4.9)$$

then for any $k < j$

$$0 = \int_{|x|=1} f(x) x_j x_k = a_{jk} \int_{|x|=1} x_j^2 x_k^2, \quad (4.10)$$

since all other terms are odd in some variable. Thus $a_{jk} = 0$ for any $k < j$. We then choose $x_n = 1$ and $x_j = 0$ for $j < n$ to get

$$\sum_{l < n} b_l = 0. \quad (4.11)$$

On the other hand, for fixed $m < n$ we choose $x_m = 1$ and $x_j = 0$ for all $j \neq m$ to get

$$(1 - n)b_m + \sum_{l < n, l \neq m} b_l = 0. \quad (4.12)$$

Hence we conclude $b_m = 0$ for all $m < n$.

We have shown that the set $\{\Phi^{jk} : (j, k) \neq (n, n)\}$ consists of $n^2 - 1$ vector fields and the dimension of its span is $\frac{1}{2}n(n-1) + \frac{1}{2}n(n-1) + (n-1) = n^2 - 1$. Thus the set is linearly independent. \square

Remark 4.2. By the definition and the divergence free condition, we have $\Phi^{nn} = -\sum_{j=1}^{n-1} \Phi^{jj}$. Therefore $\dim \text{span} \{\Phi^{jk} : 1 \leq k, j \leq n\} = n^2 - 1$.

Lemma 4.3. *Let $n \geq 2$. The flux $c_{jk} := \int_{|x|=R} \Phi^{jk} \cdot \nu$ is zero for every j, k . Here $\nu(x) = \frac{x}{|x|}$.*

Proof. Since $\text{div } \Phi^{jk}(x) = 0$ when $x \neq 0$, we have

$$c_{jk} = \int_{|x|=R} \Phi^{jk}(x-y) \cdot \nu(x) dS_x, \quad \forall |y| < R/2. \quad (4.13)$$

Choose $\phi \in C_c^\infty(\mathbb{R}^n)$ with support inside $B_{R/2}$ and $\int \phi = 1$. Then

$$\begin{aligned} c_{jk} &= \int \int_{|x|=R} \Phi^{jk}(x-y) \cdot \nu(x) dS_x \phi(y) dy \\ &= - \int_{|x|=R} \int \partial_{y_k} U_{ij}(x-y) \phi(y) dy \nu_i(x) dS_x \\ &= \int_{|x|=R} \int U_{ij}(x-y) \partial_{y_k} \phi(y) dy \nu_i(x) dS_x \\ &= \int \left(\int_{|x|=R} U_{ij}(x-y) \nu_i(x) dS_x \right) \partial_{y_k} \phi(y) dy = 0. \end{aligned} \quad (4.14)$$

We now consider Stokes and Navier-Stokes flows in the whole space and exterior domains. A *weak solution* of the Stokes system (S) (or of the Navier-Stokes system (NS)) in $\Omega \subset \mathbb{R}^n$ is a vector field $v \in W_{loc}^{1,2}(\Omega)$ that satisfied the weak form of (S) (or of (NS)) with divergence-free test functions, with no assumption on its global integrability nor its boundary value in this section.

Lemma 4.4 (Uniqueness in \mathbb{R}^n). *Let $v \in W_{loc}^{1,2}(\mathbb{R}^n)$, $n \geq 2$, be a weak solution of the Stokes system (S) in \mathbb{R}^n with zero force. If $v(x) = o(|x|)$ as $|x| \rightarrow \infty$, then v is constant.*

Proof. By the pressure independent estimates of [18] and bootstrapping, v is locally C^1 and

$$\|\nabla v\|_{L^\infty(B_R)} \leq \frac{C}{R} \|v\|_{L^\infty(B_{2R})}. \quad (4.15)$$

Taking $R \rightarrow \infty$, we get $\nabla v \equiv 0$. \square

Proposition 4.5 (Asymptotics of the Stokes flows in the whole space). *Let $n \geq 2$. Let $v \in H_{loc}^1(\mathbb{R}^n)$ be a weak solution of*

$$-\Delta v + \nabla p = f, \quad \operatorname{div} v = 0 \quad \text{in } \mathbb{R}^n \quad (4.16)$$

with f satisfying, for some $a > n + 1$,

$$|f(x)| \lesssim \langle x \rangle^{-a}, \quad \int_{\mathbb{R}^n} f(x) dx = 0. \quad (4.17)$$

Assume that v satisfies

$$|v(x)| \lesssim o(1) \quad \text{as } |x| \rightarrow \infty. \quad (4.18)$$

Then $|v(x)| \lesssim \langle x \rangle^{-n+1}$, and its asymptotics is given as

$$v_i(x) = \sum_{(j,k) \neq (n,n)} \Phi_i^{jk}(x) b_{jk} + O(\delta(x)) \quad (|x| > 1), \quad (4.19)$$

where $\delta(x) = \langle x \rangle^{-\min\{n, a-2\}}(1 + 1_{a=n+2} \log \langle x \rangle)$,

$$b_{jk} = - \int_{\mathbb{R}^n} y_k f_j(y) dy \quad \text{for } j \neq k, \quad b_{jj} = \int_{\mathbb{R}^n} (y_n f_n(y) - y_j f_j(y)) dy. \quad (4.20)$$

Proof. By uniqueness in the class (4.18) using Lemma 4.4,

$$v_i(x) = \int_{\mathbb{R}^n} U_{ij}(x-y) f_j(y) dy. \quad (4.21)$$

By (4.17),

$$\begin{aligned} v_i(x) &= \int_{\mathbb{R}^n} (U_{ij}(x-y) - U_{ij}(x)) f_j(y) dy \\ &= \sum_{j,k=1}^n \Phi_i^{jk}(x) \hat{b}_{jk} + \sum_{j,k=1}^n \int_{\mathbb{R}^n} (U_{ij}(x-y) - U_{ij}(x) + \Phi_i^{jk}(x) y_k) f_j(y) dy, \end{aligned} \quad (4.22)$$

where $\hat{b}_{jk} = - \int_{\mathbb{R}^n} y_k f_j(y) dy$ for $j, k = 1, 2, \dots, n$. By Remark 4.2 and the definition of b_{jk} ,

$$\sum_{j,k=1}^n \Phi_i^{jk}(x) \hat{b}_{jk} = \sum_{(j,k) \neq (n,n)} \Phi_i^{jk}(x) b_{jk}.$$

The second term in (4.22) is the error. To estimate it, we may assume $|x| > 2$. We split it as

$$\int_{\mathbb{R}^n} (U_{ij}(x-y) - U_{ij}(x) + \Phi_i^{jk}(x) y_k) f_j(y) dy = \int_{|y| \leq |x|/2} + \int_{|y| > |x|/2} = I + II.$$

By the Taylor theorem and the estimate $|\partial_{kl}^2 U_{ij}(x)| \lesssim |x|^{-n}$, we get for $\theta = \theta(x, y) \in [0, 1]$

$$|I| = \left| \int_{|y| \leq |x|/2} \partial_{kl}^2 U_{ij}(x - \theta y) y_k y_l f_j(y) dy \right| \lesssim |x|^{-n} \int_{|y| \leq |x|/2} \langle y \rangle^{-a+2} dy. \quad (4.23)$$

By (3.15),

$$|I| \lesssim |x|^{-\min(n, a-2)} (1 + 1_{a=n+2} \log |x|). \quad (4.24)$$

For II ,

$$|II| \lesssim \int_{|y| > |x|/2} (|x-y|^{2-n} + |x|^{2-n} + |x|^{1-n} |y|) |y|^{-a} dy = C |x|^{-a+2}. \quad (4.25)$$

The last equality is by scaling. The proof is complete. \square

We next consider the Stokes flows in exterior domains.

Proposition 4.6 (Asymptotics for the exterior Stokes flows). *Let $n \geq 2$ and $\Omega \subset \mathbb{R}^n$ be an exterior Lipschitz domain with $0 \notin \overline{\Omega}$. Assume $(v, p) \in H_{loc}^2(\overline{\Omega}) \times H_{loc}^1(\overline{\Omega})$ is a solution of*

$$-\Delta v + \nabla p = f, \quad \operatorname{div} v = 0 \quad \text{in } \Omega, \quad (4.26)$$

satisfying

$$|f(x)| \lesssim \langle x \rangle^{-a} \quad \text{with } a > n + 1, \quad (4.27)$$

$$\int_{\Omega} f + \int_{\partial\Omega} (\nu \cdot \nabla v - p\nu) = 0, \quad (4.28)$$

and

$$|v(x)| \lesssim o(1), \quad \text{as } |x| \rightarrow \infty. \quad (4.29)$$

Then its asymptotics is given as

$$v_i(x) = \tilde{b}_0 H_i(x) + \sum_{(j,k) \neq (n,n)} \Phi_i^{jk}(x) \tilde{b}_{jk} + O(\delta(x)), \quad (4.30)$$

where $\delta(x) = \langle x \rangle^{-\min\{n, a-2\}}(1 + 1_{a=n+2} \log \langle x \rangle)$,

$$\tilde{b}_0 = \int_{\partial\Omega} v \cdot \nu dS, \quad H(x) = \nabla E(x) = \frac{-x}{n\omega_n |x|^n}, \quad (4.31)$$

and

$$\tilde{b}_{jk} = \begin{cases} -\int_{\Omega} y_k f_j dy + \int_{\partial\Omega} (v_j \nu_k - y_k \nabla v_j \cdot \nu + y_k p \nu_j) dS & \text{if } j \neq k, \\ \int_{\Omega} (y_n f_n - y_j f_j) dy + \int_{\partial\Omega} \{(v_j \nu_j - y_j \nabla v_j \cdot \nu + y_j p \nu_j)\} dS & \text{if } j = k. \\ -\int_{\partial\Omega} \{(v_n \nu_n - y_n \nabla v_n \cdot \nu + y_n p \nu_n)\} dS & \end{cases} \quad (4.32)$$

Remark 4.7. (i) From the proof of Lemma 4.3, we see that H is linearly independent of the vectors Φ^{jk} for $1 \leq j, k \leq n$.

(ii) If we restrict a solution (v, p) of Proposition 4.5 to Ω , since

$$\int_{\Omega^c} f = \int_{\Omega^c} -\Delta v_i + \partial_i p = \int_{\partial\Omega} \partial_j v_i \nu_j - p \nu_i, \quad (4.33)$$

we get condition (4.28) from $\int_{\mathbb{R}^n} f = 0$.

Proof. First note that, by replacing v by $\tilde{v} = v - \tilde{b}_0 H$, we may assume $\int_{\partial\Omega} v \cdot \nu = 0$. Note that (4.28) and (4.32) are not changed by this replacement because, for (4.28),

$$\int_{\partial\Omega} \nu \cdot \nabla H_i = \int_{\Omega \cap B_R} \operatorname{div} \nabla H_i - \int_{\partial B_R} \frac{x}{R} \cdot \nabla H_i = O(1/R), \quad (4.34)$$

which vanishes as $R \rightarrow \infty$; For (4.32) and $R > \operatorname{diam}(\Omega)$,

$$\begin{aligned} \int_{\partial\Omega} \{H_j \nu_k - y_k \partial_l H_j \nu_l\} &= \int_{\partial\Omega} \{2H_j \nu_k - \partial_l (y_k H_j) \nu_l\} \\ &= \int_{\Omega \cap B_R} \{2\partial_k H_j - \Delta(y_k H_j)\} - \int_{\partial B_R} \{H_j \nu_k - y_k \partial_l H_j \nu_l\} = 0 + \delta_{kj}. \end{aligned} \quad (4.35)$$

Let Ω_1 be any exterior domain with $\overline{\Omega}_1 \subset \Omega$, and χ be any smooth function with $\text{supp } \chi \subset \Omega$ and $\chi = 1$ in Ω_1 . We define (w, q) by

$$w = v\chi + \hat{v}, \quad q = p\chi, \quad (4.36)$$

where \hat{v} is a solution of $\text{div } \hat{v} = -v \cdot \nabla \chi$ in \mathbb{R}^n . Thanks to the condition $\int_{\partial\Omega} v \cdot \nu = 0$, we can choose \hat{v} satisfying $\text{supp } \hat{v} \subset \overline{\Omega} \setminus \Omega_1$, $\|\nabla \hat{v}\|_{L^s} \lesssim_s \|v \cdot \nabla \chi\|_{L^s}$ by [5, Theorem III.3.1]. Then (w, q) satisfies

$$\begin{aligned} -\Delta w + \nabla q &= g, \quad \text{div } w = 0 \quad \text{in } \mathbb{R}^n, \\ g &= f\chi - \partial_l(v\partial_l\chi) - \partial_lv\partial_l\chi - \Delta\hat{v} + p\nabla\chi. \end{aligned}$$

From the assumption for f , we easily see that $|g(x)| \lesssim \langle x \rangle^{-a}$ and $\int_{\mathbb{R}^n} g dx = 0$ because

$$\begin{aligned} \int_{\mathbb{R}^n} g dx &= \int_{\Omega} \{f\chi - \partial_l(v\partial_l\chi) - \partial_lv\partial_l(\chi - 1) - \Delta\hat{v} + p\nabla(\chi - 1)\} \\ &= \int_{\Omega} f\chi - \int_{\partial\Omega} v\partial_l\chi\nu_l + \int_{\Omega} \Delta v(\chi - 1) + \int_{\partial\Omega} \partial_lv\nu_l - \int_{\Omega} \nabla p(\chi - 1) - \int_{\partial\Omega} p\nu \\ &= \int_{\Omega} f + \int_{\partial\Omega} (\nu \cdot \nabla v - p\nu), \end{aligned}$$

which is zero by assumption (4.28). Then Proposition 4.5 shows $w_i = \Phi_i^{jk} b_{jk} + R_i$ with $|R_i(x)| \leq C\delta(x)$ and

$$b_{jk} = \begin{cases} -\int_{\mathbb{R}^n} y_k g_j(y) dy & \text{for } j \neq k, \\ -\int_{\mathbb{R}^n} (y_k g_j(y) - y_n g_n(y)) dy & \text{for } j = k. \end{cases} \quad (4.37)$$

We claim that b_{jk} is independent of the choice of the cut-off (4.36). Indeed, let χ' , (w', q') be another cut-off solution of (4.36), and b'_{jk} be as in (4.37), then for $j \neq k$,

$$\begin{aligned} b_{jk} - b'_{jk} &= -\int_{\mathbb{R}^n} y_k \{-\Delta(v_j(\chi - \chi') + \hat{v}_j - \hat{v}'_j) + \partial_j(p(\chi - \chi'))\} \\ &= \int_{\mathbb{R}^n} \partial_j y_k (p(\chi - \chi')) = 0. \end{aligned}$$

Similary,

$$b_{jj} - b'_{jj} = \int_{\mathbb{R}^n} \{\partial_j y_j (p(\chi - \chi')) - \partial_n y_n (p(\chi - \chi'))\} = 0.$$

Thus the claim follows.

Now consider a sequence of cut-off functions χ_m , $m = 1, 2, \dots$, such that $0 \leq \chi_m(x) \leq \chi_{m+1}(x) \leq 1$, $(m = 1, 2, \dots)$ and $\chi_m(x) \rightarrow 1$ as $m \rightarrow \infty$ for all $x \in \Omega$, and choose $(w^{(m)}, q^{(m)})$, $b_{jk}^{(m)}$ as in (4.36), (4.37). Since $b_{jk}^{(m)}$ is independent of m , it suffices to prove $\lim_{m \rightarrow \infty} b_{jk}^{(m)} = \tilde{b}_{jk}$. We only consider the case $j \neq k$, since the case $j = k$ is shown in the same way. We divide

$$\begin{aligned} b_{jk}^{(m)} &= -\int_{\mathbb{R}^n} y_k \{f_j\chi - \partial_l(v_j\partial_l\chi) - \partial_lv_j\partial_l\chi - \Delta\hat{v}_j + p\partial_j\chi\} dy \\ &= I + II + III + IV + V. \end{aligned}$$

Then it easily follows that $I \rightarrow -\int_{\Omega} y_k f_j dy$ as $m \rightarrow \infty$ and that $IV = 0$. By integration by parts, we also observe

$$\begin{aligned} II &= -\int_{\mathbb{R}^n} \partial_l(y_k) v_j \partial_l \chi dy = -\int_{\Omega} v_j \partial_k(\chi - 1) dy \\ &= \int_{\Omega} \partial_k v_j(\chi - 1) + \int_{\partial\Omega} v_j \nu_k \rightarrow \int_{\partial\Omega} v_j \nu_k, \\ III &= -\int_{\Omega} \partial_l(y_k \partial_j v_l)(\chi - 1) dy - \int_{\partial\Omega} y_k \partial_l v_j \nu_l \rightarrow -\int_{\partial\Omega} y_k \partial_l v_j \nu_l, \\ V &= \int_{\Omega} \partial_j(y_k p)(\chi - 1) dy + \int_{\partial\Omega} y_k p \nu_j \rightarrow \int_{\partial\Omega} y_k p \nu_j, \end{aligned}$$

as $m \rightarrow \infty$, using $\chi \rightarrow 1$ in Ω . Thus we have proved (4.32). \square

Theorem 4.8 (Asymptotics of fast decaying Navier-Stokes flows in \mathbb{R}^n). *Let $n \geq 3$, and $u \in H_{loc}^1(\mathbb{R}^n)$ be a weak solution of*

$$-\Delta u + (u \cdot \nabla)u + \nabla p = f, \quad \operatorname{div} u = 0 \quad \text{in } \mathbb{R}^n. \quad (4.38)$$

There exists $\varepsilon_0 > 0$ such that if for some $\varepsilon \in (0, \varepsilon_0]$,

$$|f(x)| \leq \varepsilon \langle x \rangle^{-a} \quad \text{with } a > n + 1, \quad \int_{\mathbb{R}^n} f(x) dx = 0, \quad (4.39)$$

and

$$|u(x)| \leq \varepsilon \langle x \rangle^{1-n}, \quad (4.40)$$

then its asymptotics is given as

$$u_i(x) = \sum_{(k,j) \neq (n,n)} \Phi_i^{jk}(x) a_{jk} + O(\varepsilon \delta(x)), \quad (4.41)$$

where $\delta(x) = \langle x \rangle^{-\min\{a-2, n\}}(1 + 1_{a=n+2} \log \langle x \rangle)$,

$$a_{jk} = \begin{cases} b_{jk} - \int_{\mathbb{R}^n} u_j u_k dy & \text{for } j \neq k, \\ b_{jj} - \int_{\mathbb{R}^n} (u_j^2 - u_n^2) dy & \text{for } j = k. \end{cases}$$

Here b_{ij} are the constants given by (4.20).

Remark. We can replace (4.40) by a weaker condition $|u(x)| \leq \varepsilon \langle x \rangle^{-1-\sigma}$, $\sigma > 0$: Under this weaker condition, we can improve the decay iteratively, $|u(x)| \lesssim \varepsilon \langle x \rangle^{-1-(1+k/2)\sigma}$, $k \in \mathbb{N}$, as in the proof of Theorem 3.8, Case (ii).

Proof. By the scaling and the bootstrapping argument as in [18], we see $|\nabla u(x)| \lesssim \varepsilon \langle x \rangle^{-n}$. Indeed, if $R = \frac{1}{2}|x_0| > 2$, let $v(y) = R^{n-1}u(x)$, $\pi(y) = R^n p(x)$ and $g(y) = R^{n+1}f(x)$ with $x = x_0 + Ry$. Then v satisfies

$$-\Delta v + R^{2-n}v \cdot \nabla v + \nabla \pi = g \quad (4.42)$$

with $|v| \lesssim \varepsilon$ and $g \lesssim \varepsilon$ in B_1 . By bootstrapping (using the pressure-independent Stokes estimate of [18]), one gets $|\nabla v| \lesssim \varepsilon$ in $B_{1/2}$, which implies $|\nabla u(x_0)| \lesssim \varepsilon R^{-n}$.

Note that u is the solution of the Stokes equations with force $\tilde{f} = f - u \cdot \nabla u$ satisfying $|\tilde{f}(x)| \lesssim \langle x \rangle^{-\min\{a, 2n-1\}}$. Since $\min\{a, 2n-1\} > n+1$ and $2n-1 > n+2$ using $n \geq 3$, it follows from Proposition 4.5 that

$$u_i(x) = \sum_{(j,k) \neq (n,n)} \Phi_i^{jk}(x) a_{jk} + O(\delta(x)),$$

where

$$a_{jk} = \begin{cases} -\int_{\mathbb{R}^n} y_k(f_j - u \cdot \nabla u_j)(y) dy & \text{for } j \neq k, \\ -\int_{\mathbb{R}^n} (y_j f_j - y_n f_n) - (y_j u \cdot \nabla u_j - y_n u \cdot \nabla u_n) dy & \text{for } j = k. \end{cases}$$

Then noting that $\int_{\mathbb{R}^n} y_k(u \cdot \nabla u_j)(y) dy = -\int_{\mathbb{R}^n} u_j u_k(y) dy$, we obtain the desired result. \square

Theorem 4.9 (Asymptotics of fast decaying exterior Navier-Stokes flows). *Let $n \geq 3$ and $\Omega \subset \mathbb{R}^n$ be an exterior Lipschitz domain with $0 \notin \overline{\Omega}$, and let $(u, p) \in H_{loc}^2(\overline{\Omega}) \times H_{loc}^1(\overline{\Omega})$ be a solution of*

$$-\Delta u + (u \cdot \nabla)u + \nabla p = f, \quad \operatorname{div} u = 0 \quad \text{in } \Omega. \quad (4.43)$$

There exists $\varepsilon_0 > 0$ such that if for some $\varepsilon \in (0, \varepsilon_0]$, $|f(x)| \lesssim \varepsilon \langle x \rangle^{-a}$ with $a > n + 1$,

$$|u(x)| \lesssim \varepsilon \langle x \rangle^{-n+1}, \quad (4.44)$$

$$\int_{\Omega} f + \int_{\partial\Omega} (\nu \cdot \nabla u - p\nu - (u \cdot \nu)u) = 0. \quad (4.45)$$

Then its asymptotics is given as

$$u_i(x) = \tilde{b}_0 H_i(x) + \sum_{(j,k) \neq (n,n)} \Phi_i^{jk}(x) \tilde{a}_{jk} + O(\varepsilon \delta(x)), \quad (4.46)$$

where $\delta(x) = \langle x \rangle^{-\min\{a-2, n\}}(1 + 1_{a=n+2} \log \langle x \rangle)$,

$$\tilde{a}_{jk} = \begin{cases} \tilde{b}_{jk} - \int_{\Omega} u_j u_k dy + \int_{\partial\Omega} y_k u_j u \cdot \nu dS & \text{for } j \neq k, \\ \tilde{b}_{jj} - \int_{\Omega} (u_j^2 - u_n^2) dy + \int_{\partial\Omega} (y_j u_j - y_n u_n) u \cdot \nu dS & \text{for } j = k. \end{cases}$$

Above \tilde{b}_0 , \tilde{b}_{jk} and $H(x)$ are defined in Proposition 4.6.

Remark 4.10. If we restrict a solution (u, p) of Theorem 4.8 to Ω , since

$$\int_{\Omega^c} f = \int_{\Omega^c} -\Delta u_i + \partial_j(u_j u_i) + \partial_i p = \int_{\partial\Omega} (\partial_j v_i - u_j u_i) \nu_j - p \nu_i, \quad (4.47)$$

we get condition (4.45) from $\int_{\mathbb{R}^n} f = 0$.

Proof. As in the proof of Theorem 4.8, (u, p) satisfies the linear Stokes system with the force $\tilde{f} = f - (u \cdot \nabla)u$ in Ω and $|\nabla u(x)| \lesssim \varepsilon |x|^{-n}$. Here $|\tilde{f}(x)| \leq \varepsilon \langle x \rangle^{-\min\{a, 2n-1\}}$ with $\min\{a, 2n-1\} > n + 1$. (4.45) and the integration by parts yield

$$\int_{\Omega} \tilde{f} + \int_{\partial\Omega} (\nu \cdot \nabla u - p\nu) = 0. \quad (4.48)$$

Then Proposition 4.6 shows

$$u(x) = \tilde{b}_0 H(x) + \sum_{k=1}^n \Phi^{jk}(x) \tilde{a}_{jk} + O(\varepsilon \delta(x)).$$

Here we have for $j \neq k$ that

$$\begin{aligned} \tilde{a}_{jk} &= -\int_{\Omega} y_k(f_j(y) - u \cdot \nabla u_j(y)) dy + \int_{\partial\Omega} (u_j \nu_k - y_k \nabla u_j \cdot \nu + y_k p \nu_j) dS \\ &= -\int_{\Omega} (y_k f_j(y) + u_j u_k(y)) dy + \int_{\partial\Omega} (u_j \nu_k - y_k \nabla u_j \cdot \nu + y_k p \nu_j + y_k u_j u \cdot \nu) dS. \end{aligned} \quad (4.49)$$

The case $j = k$ is handled in the same way. Hence the proof is complete. \square

5 Appendix: Axisymmetric self-similar solutions in \mathbb{R}_+^3

In this appendix we consider the nonexistence of minus one homogeneous solutions of the steady-state Navier-Stokes equations in the half-space \mathbb{R}_+^3 with the *Navier* boundary conditions (BC),

$$-\Delta u + (u \cdot \nabla)u + \nabla p = 0, \quad \text{in } \mathbb{R}_+^3, \quad (5.1)$$

$$u \cdot \nu = 0, \quad \left((1 - \gamma) \frac{\partial u}{\partial \nu} + \gamma u \right) \times \nu = 0, \quad \text{on } \partial \mathbb{R}_+^3 \setminus \{0\}, \quad (5.2)$$

for some given $\gamma \in [0, 1]$, and ν is the unit outernormal, $\nu = (0, 0, -1)$ for \mathbb{R}_+^3 . Note that the Navier BC becomes the zero Dirichlet (no-slip) BC if $\gamma = 1$, which is what we used in Section 3. When $\gamma = 0$, it agrees with the *slip BC* for a half space, see e.g. [20]. Their nonexistence excludes an obstacle for proving the asymptotic results in Section 3 which have faster decay, even under the more general Navier BC. Recall that in the whole space we have the family of Slezkin-Landau solutions.

Theorem 5.1. *Let u be a minus one homogeneous solution of the Navier-Stokes equations (5.1) in \mathbb{R}_+^3 with the Navier BC (5.2) for some $\gamma \in [0, 1]$. If u is axially symmetric with respect to x_3 -axis, then u vanishes.*

Remark. Note u is allowed to have nonzero u_θ -component. If we do not impose any BC, then the restrictions of the Slezkin-Landau solutions are non-trivial solutions.

The following proof is adapted from the corresponding argument of Tsai and Sverak for the whole space [19, Section 4.3].

Proof. We use the spherical coordinate (ρ, θ, φ) , where $\rho = |x|$, θ is azimuthal angle, and φ is between the angle x and x_3 -axis. The axially symmetric solution is of the form

$$u = \frac{1}{\rho} f(\varphi) e_\rho + \frac{1}{\rho} g(\varphi) e_\varphi + \frac{1}{\rho} h(\varphi) e_\theta. \quad (5.3)$$

We note that the boundary conditions (5.2) becomes

$$g\left(\frac{\pi}{2}\right) = 0, \quad (1 - \gamma)f'\left(\frac{\pi}{2}\right) + \gamma f\left(\frac{\pi}{2}\right) = 0, \quad (1 - \gamma)h'\left(\frac{\pi}{2}\right) + \gamma h\left(\frac{\pi}{2}\right) = 0. \quad (5.4)$$

We also observe, due to symmetry, that

$$f'(0) = g(0) = h(0) = 0. \quad (5.5)$$

The equations can be rewritten in spherical coordinates as follows:

$$f'' + f' \cot \varphi = g f' - (f^2 + g^2) - 2p, \quad (5.6)$$

$$f' = g g' + p', \quad (5.7)$$

$$(h' + h \cot \varphi)' = g(h' + h \cot \varphi). \quad (5.8)$$

$$f + g' + g \cot \varphi = 0. \quad (5.9)$$

Setting $H(\varphi) := h' + h \cot \varphi = (h \sin \varphi)' / \sin \varphi$, we see that (5.8) is rewritten as $H' = gH$. We claim that $H = 0$, which obviously implies $h = 0$, due to boundary conditions (5.24). We treat the cases of $\gamma = 0$, $0 < \gamma < 1$ and $\gamma = 1$, separately. We note first that if H has a zero at a point in $[0, \pi/2]$, it vanishes everywhere due to uniqueness of ODE. In case that $\gamma = 0$, it is direct via (5.4) that $H(\pi/2) = 0$, and thus $H = 0$. In case that $0 < \gamma < 1$, we note that

$$H\left(\frac{\pi}{2}\right) = h'\left(\frac{\pi}{2}\right) = -\frac{\gamma}{1 - \gamma} h\left(\frac{\pi}{2}\right). \quad (5.10)$$

On the other hand, we also observe that

$$\int_0^{\frac{\pi}{2}} H(\varphi) \sin \varphi d\varphi = h\left(\frac{\pi}{2}\right). \quad (5.11)$$

Suppose that H has no zero, which means that H is either positive or negative on $[0, \pi/2]$. If H is positive, then $h(\pi/2) < 0$ via (5.10). Then, it is contrary to (5.11). The other case that H is negative also lead to a contradiction. Thus, $H = 0$. Finally, for the case $\gamma = 0$, we have due to (5.24)

$$\int_0^{\frac{\pi}{2}} H(\varphi) \sin \varphi d\varphi = h\left(\frac{\pi}{2}\right) = 0. \quad (5.12)$$

This implies that H has a zero. Therefore, we conclude that H vanishes, and so $h = 0$.

Integrating (5.7), we have

$$f = \frac{g^2}{2} + p + C_1 \quad (5.13)$$

for some constant C_1 . Combining (5.6) and (5.13), we obtain

$$f'' + f' \cot \varphi = g f' - f^2 - 2f + 2C_1. \quad (5.14)$$

We set $A := f(\varphi) \sin \varphi$ and $B := g(\varphi) \sin \varphi$. Noting that $-A = B'$, we see that (5.14) becomes

$$(f' \sin \varphi)' = (Bf)' + 2B' + 2C_1 \sin \varphi. \quad (5.15)$$

Therefore, we obtain

$$f' \sin \varphi = Bf + 2B - 2C_1 \cos \varphi + C_2. \quad (5.16)$$

Via (5.5), we see that $C_2 = 2C_1$ and thus,

$$f' \sin \varphi = Bf + 2B - 2C_1(\cos \varphi - 1). \quad (5.17)$$

Let $L(t) = B(\varphi)$ with $t = \cos \varphi$. Noting that $B' = -L' \sin \varphi$ and $B'' = L'' \sin^2 \varphi - L' \cos \varphi$, we observe that L satisfies

$$(1 - t^2)L'' + 2L + LL' = 2C_1(t - 1), \quad L(0) = L(1) = 0. \quad (5.18)$$

Using the change of variable $L(t) := (1 - t^2)v(t)$, we see that v solves

$$\left((1 - t^2)^2 v'\right)' + \left(\frac{(1 - t^2)^2 v^2}{2}\right)' - 2C_1(t - 1) = 0, \quad v(0) = 0, \quad (5.19)$$

which can be simplified as follows:

$$v' + \frac{v^2}{2} = \frac{C_1}{(1 + t)^2} + \frac{C_3}{(1 - t^2)^2}, \quad v(0) = 0. \quad (5.20)$$

Since v is bounded over $t \in [0, 1]$, we see that $C_3 = 0$, and therefore, v satisfies

$$v' + \frac{v^2}{2} = \frac{C_1}{(1 + t)^2}, \quad v(0) = 0. \quad (5.21)$$

In addition, we note that

$$v'(0) = -g'\left(\frac{\pi}{2}\right) = C_1, \quad v''(0) = g''\left(\frac{\pi}{2}\right) = -2C_1, \quad (5.22)$$

Recalling that $f = -g' - g \cot \varphi$ and by taking one more derivative, we also see that $f' = -g'' - g' \cot \varphi + \frac{g}{\sin^2 \varphi}$. Therefore, we obtain

$$f\left(\frac{\pi}{2}\right) = -g'\left(\frac{\pi}{2}\right), \quad f'\left(\frac{\pi}{2}\right) = -g''\left(\frac{\pi}{2}\right). \quad (5.23)$$

Combining (5.4) and (5.23), we see that

$$0 = (1 - \gamma)f'\left(\frac{\pi}{2}\right) + \gamma f\left(\frac{\pi}{2}\right) = 2(1 - \gamma)C_1 + \gamma C_1 = (2 - \gamma)C_1. \quad (5.24)$$

Since $\gamma \in [0, 1]$, we conclude that $C_1 = 0$, which implies $v = 0$. Then, it is straightforward that $f = g = 0$, which implies that u vanishes. This completes the proof. \square

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